Slit-slide-sew bijections for oriented planar maps

(slĭt) (slīd) (sō)

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joint with Éric Fusy, Baptiste Louf

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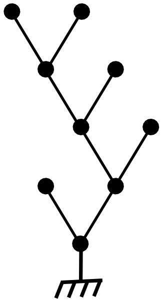
Bijectively growing stuff

Example: binary trees [Rémy '85].



$\mathcal{A}_n$: rooted binary plane trees with

- n nodes
- $n+1$ leaves
- $2n+1$ edges (counting the root)

$$|\mathcal{A}_n| = \frac{1}{n+1}\binom{2n}{n}$$

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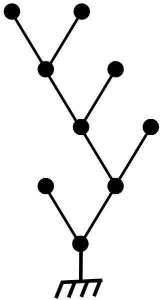
$$2\,(2n+1)\,|\mathcal{A}_n| = (n+2)\,|\mathcal{A}_{n+1}|$$



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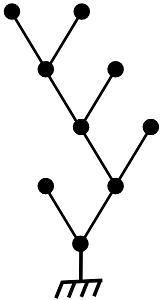

$$2\,\underbrace{(2n+1)}_{\text{edge}}\,|\mathcal{A}_n| = (n+2)\,|\mathcal{A}_{n+1}|$$

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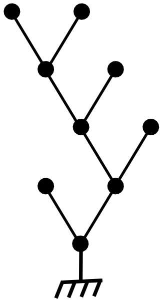

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mark *left* or *right*

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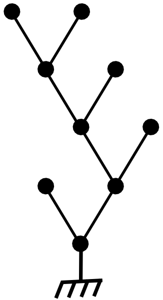

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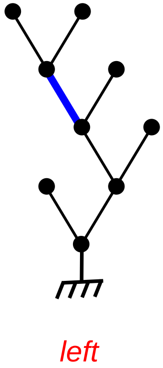

- ☐ Binary tree with a distinguished edge and a mark *left* or *right*.

Bijectively growing stuff

Example: binary trees [Rémy ’85].

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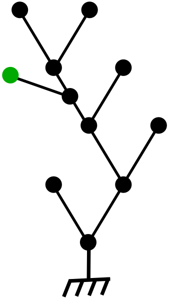

- Binary tree with a distinguished edge and a mark *left or right*.
- We add a new edge to the left or right of the distinguished edge and distinguish the leaf at its extremity.

Bijectively growing stuff

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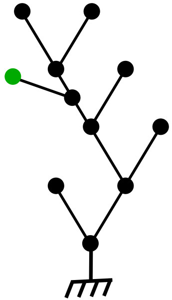

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Trees [Marckert '22]

Constellations [B.–Korkotashvili '24]

Maps [B. '14, B. '19, Louf '19, Schabanel '25]

What is a map?

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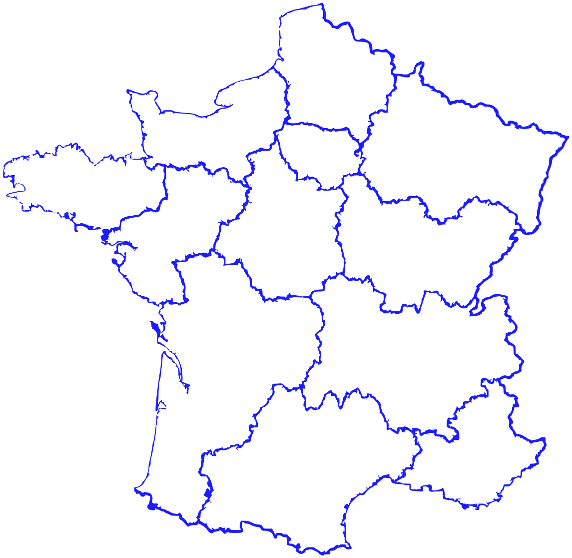

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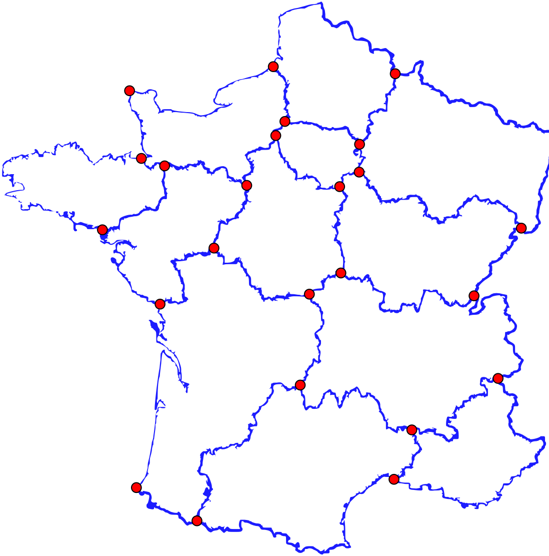

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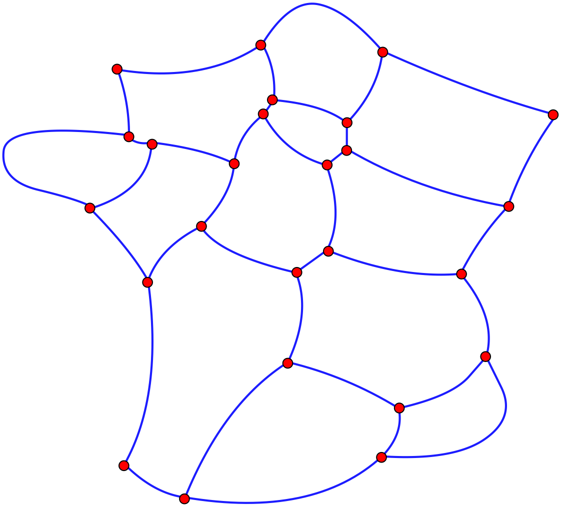

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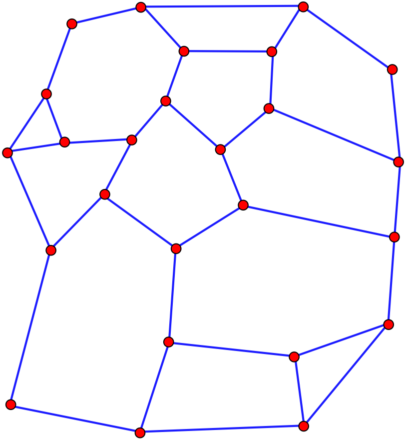
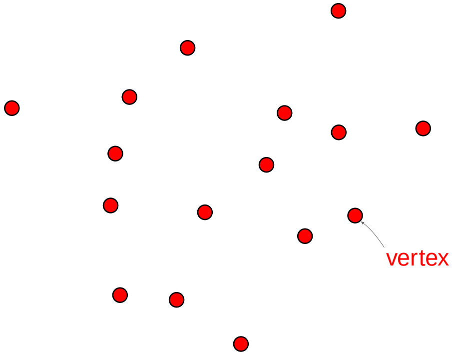


Plane map, formally

- We have **vertices**



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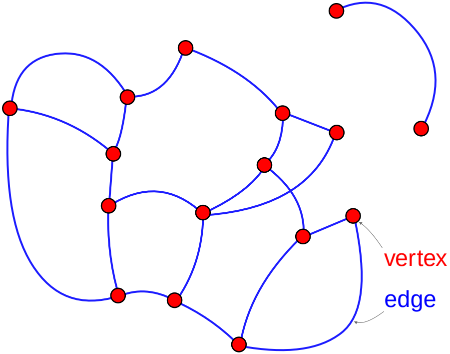

- We have **vertices**
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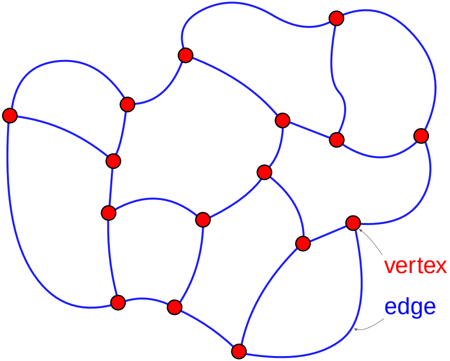

- We have **vertices**
- linked by **edges**
  - without crossings;
  - in a connected way.

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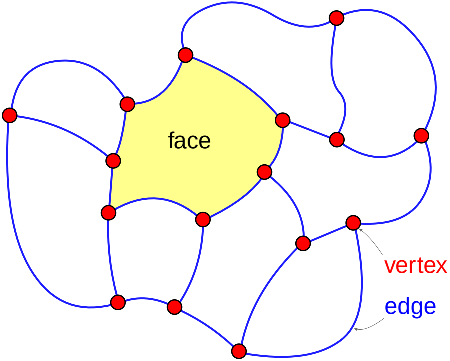

- We have **vertices**
- linked by **edges**
  - without crossings;
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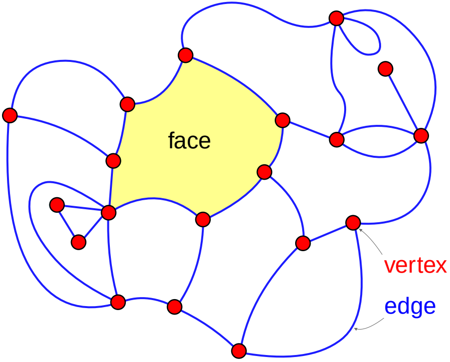
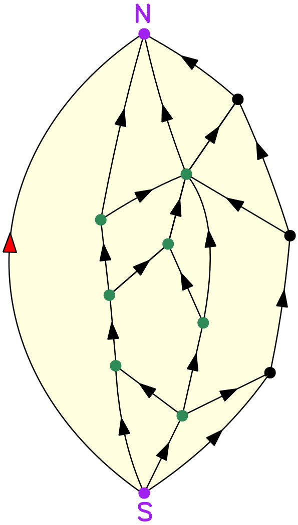
- Delimited areas are **faces**.

Plane map, formally



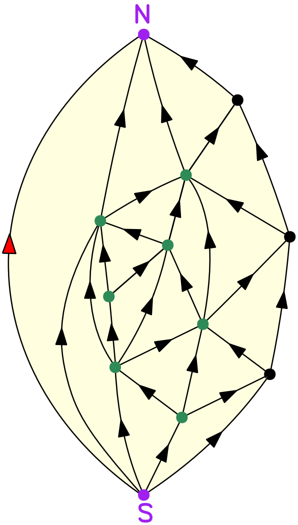
- We have **vertices**
- linked by **edges**
  - without crossings;
  - in a connected way.
- Delimited areas are **faces**.
- Multiple edges and loops are allowed.

Bipolar oriented map



- ☐ One source: the South Pole $\mathbb{S}$.
- ☐ One sink: the North Pole $\mathbb{N}$.
- ☐ The root links $\mathbb{S}$ to $\mathbb{N}$.
- ☐ No directed cycles.

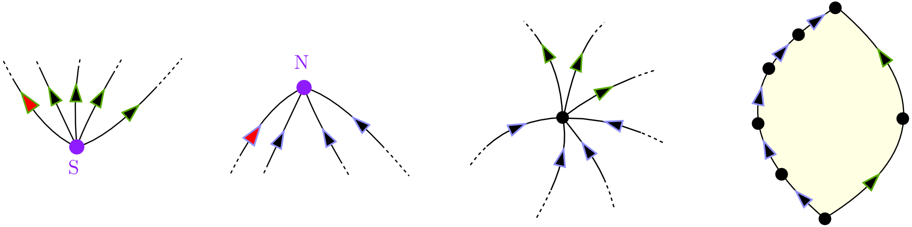
Bipolar oriented quasi-triangulation



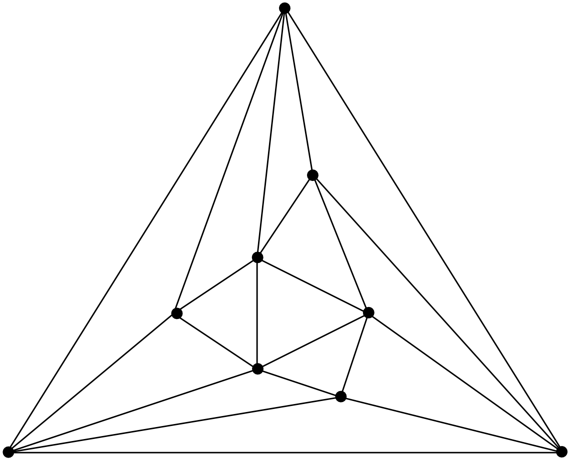
- One source: the South Pole $\mathbb{S}$.
- One sink: the North Pole $\mathbb{N}$.
- The root links $\mathbb{S}$ to $\mathbb{N}$.
- No directed cycles.
- Internal faces have degree 3.

Local rules

Bipolar oriented maps are characterized by these local rules:

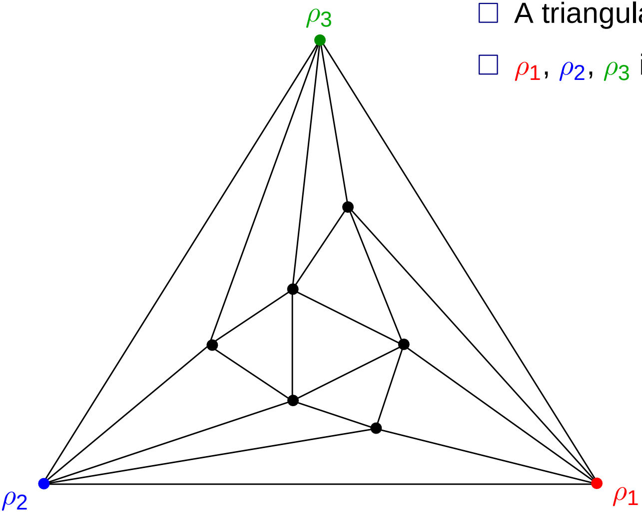


Schnyder woods



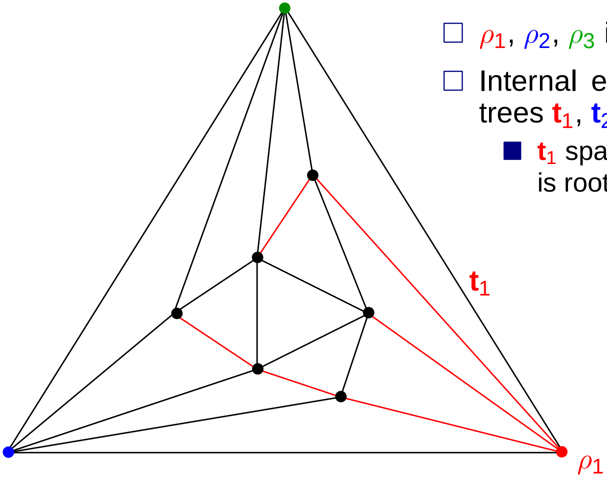
- A triangulation.

Schnyder woods



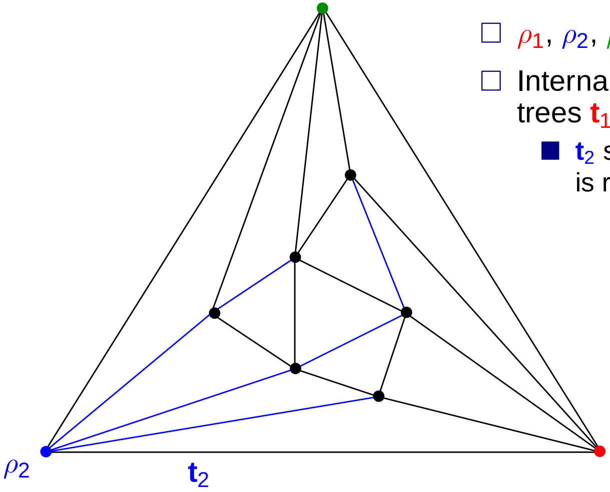
- A triangulation.
- ρ_1, ρ_2, ρ_3 in clockwise order.

Schnyder woods



- A triangulation.
- ρ_1, ρ_2, ρ_3 in clockwise order.
- Internal edges partitioned into trees $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$:
  - $\mathbf{t}_1$ spans internal vertices and is rooted at ρ_1;

Schnyder woods



- A triangulation.
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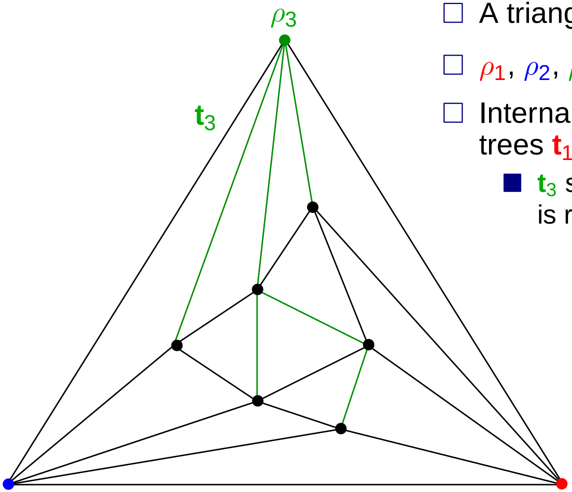
- Internal edges partitioned into trees $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$:
  - $\mathbf{t}_2$ spans internal vertices and is rooted at ρ_2;

Schnyder woods



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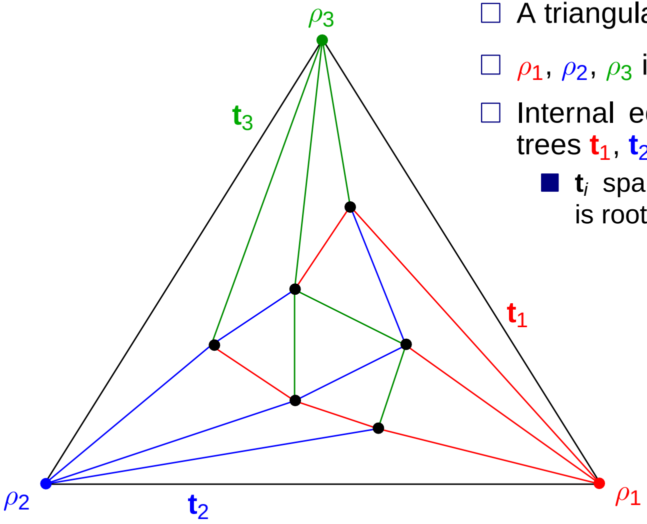

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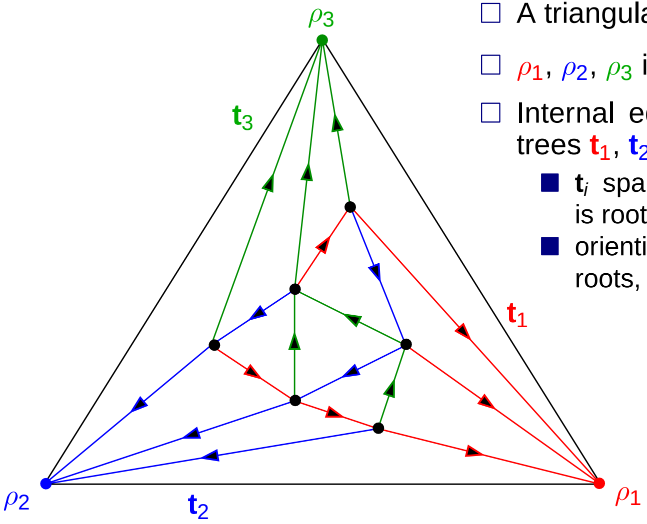

- A triangulation.
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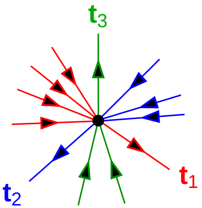
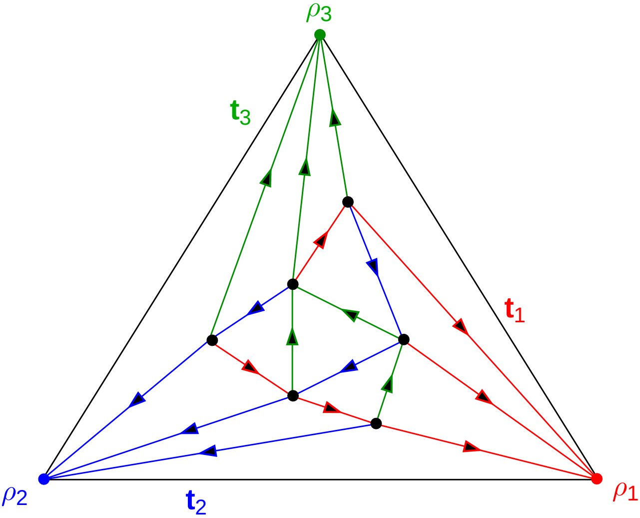
- Internal edges partitioned into trees $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{t}_3$:
  - $\mathbf{t}_i$ spans internal vertices and is rooted at ρ_i ;
  - orienting trees toward their roots, one has the local rule



Schnyder woods as b.o.m.s



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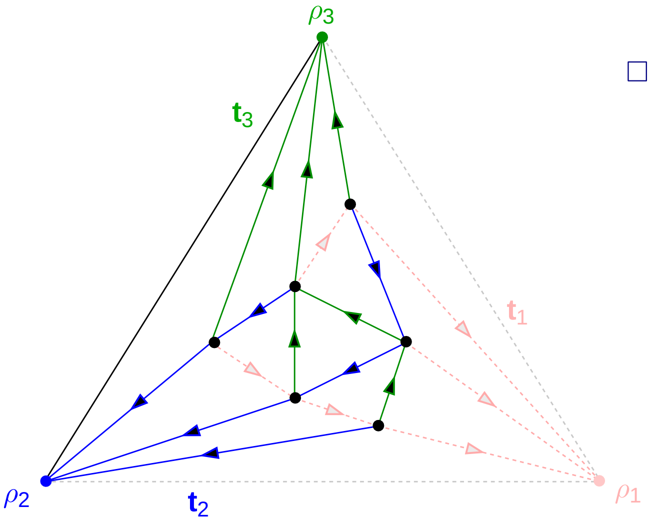
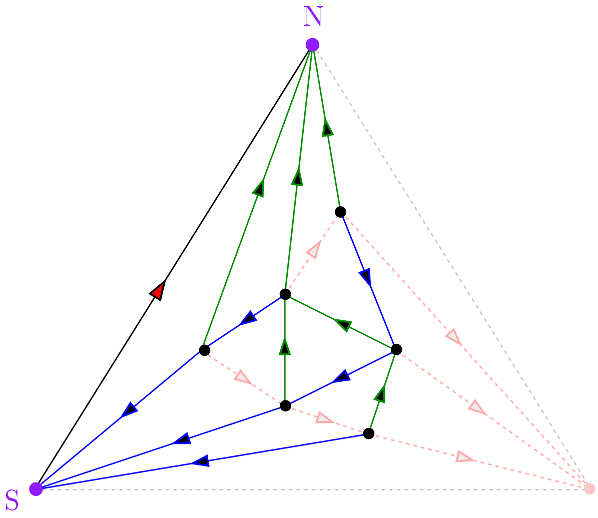


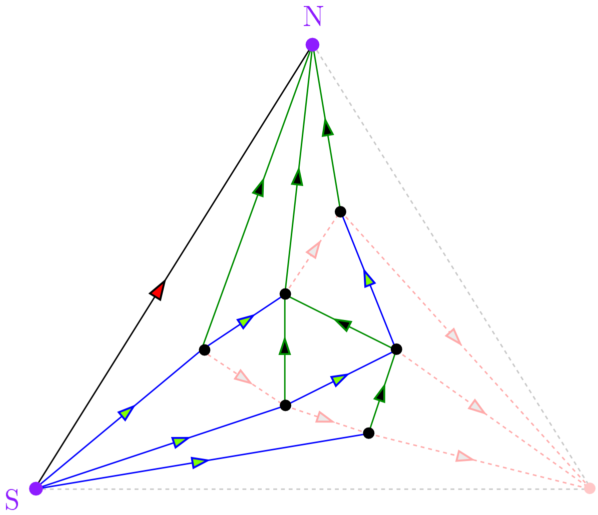
- Remove $\mathbf{t}_1$, ρ_1 and the two incident external edges.

Schnyder woods as b.o.m.s



- Remove $\mathbf{t}_1$, ρ_1 and the two incident external edges.
- Set $\mathrm{S} := \rho_2$, $\mathrm{N} := \rho_3$, and the root.

Schnyder woods as b.o.m.s



- Remove $\mathbf{t}_1$, ρ_1 and the two incident external edges.
- Set $\mathrm{S} := \rho_2$, $\mathrm{N} := \rho_3$, and the root.
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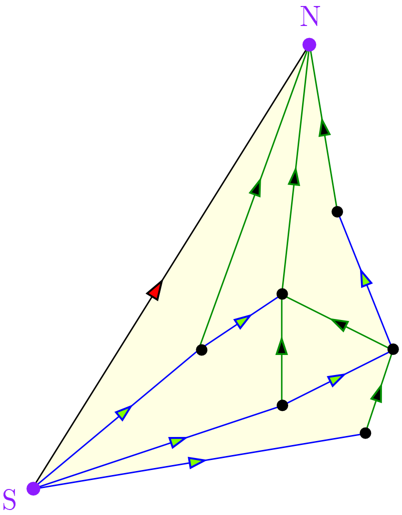

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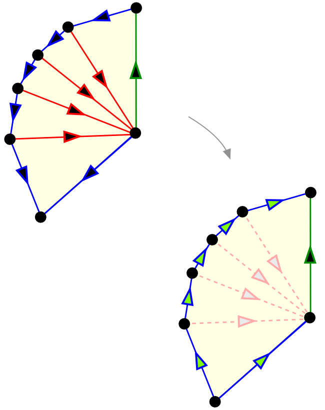

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Internal faces have right length 2.

Let's count

$k \geq 0$ internal vertices external face degree $j \geq 2$

Bipolar oriented quasi-triangulations

$$T_{k,j} = j\,(j-1)\frac{(3k+2j-4)!}{k!\,(k+j-1)!\,(k+j)!}$$

Bipolar oriented maps

$\ell \geq 1$ internal faces

$$B_{k,\ell,j} = j\,(j-1)\frac{(k+\ell-2)!\,(k+\ell+j-2)!\,(k+\ell+j-3)!}{k!\,(k+j)!\,(k+j-1)!\,\ell!\,(\ell-1)!\,(\ell-2)!}$$

Bipolar oriented maps with internal faces of right length 2

$$S_{k,j} = j\,(j-1)\,(j-2)\frac{(2k+2j-4)!\,(2k+j-3)!}{k!\,(k+j)!\,(k+j-1)!\,(k+j-2)!}$$

1& 2: recursive decomposition [Bousquet-Mélou '11]

3: bijective method [Bernardi–Bonichon '09]

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Schnyder woods on $k+j+1$ vertices, with ρ_1 of degree j

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1& 2: recursive decomposition [Bousquet-Mélou '11]
3: bijective method [Bernardi–Bonichon '09]

Combinatorial identities

Bipolar oriented quasi-triangulations $k \geq 1, j \geq 2$

$$k\, T_{k,j} = \left(1 - \frac{2}{j+1}\right)(3k + 2j - 4)\, T_{k-1,j+1}$$

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Schnyder woods $k \geq 1, j \geq 3$

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internal vertex

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edge?

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internal vertex

edge?

$1 - \frac{?}{\text{external face degree}}$

Slit, slide, sew

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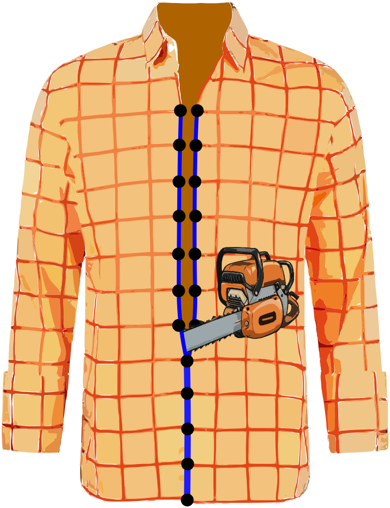

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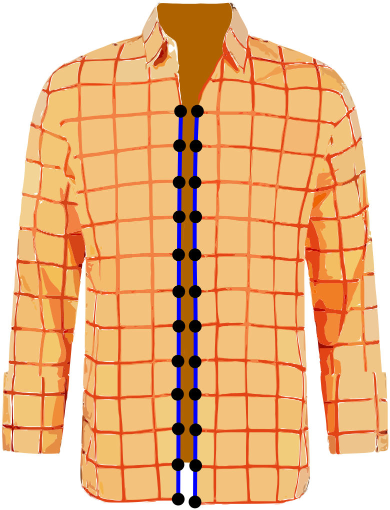

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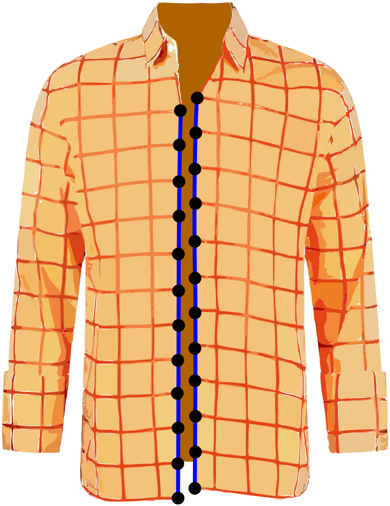

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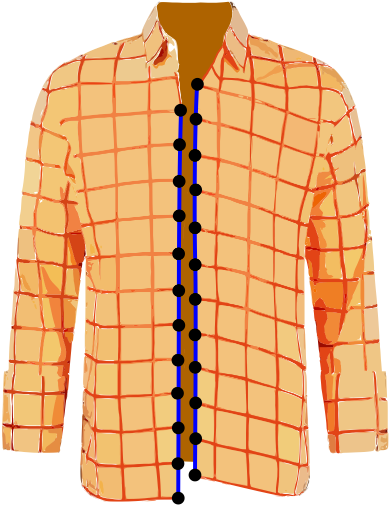

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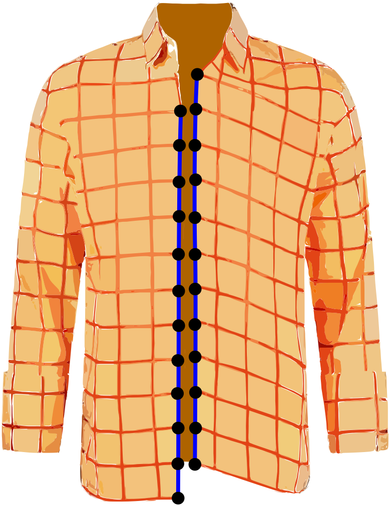

Slit, slide, sew

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Example: degree transfer in a bip. or quasibip. map

maps with r (rooted) faces $f_1, \ldots, f_r$, of degrees given by

$\mathbf{a} = (a_1, \ldots, a_r)$ or $\tilde{\mathbf{a}} := (a_1 + 1, a_2, \ldots, a_r, a_r - 1)$

$$(a_1 + 1) \lfloor a_r/2 \rfloor M(\mathbf{a}) = \lfloor \tilde{a}_1/2 \rfloor (\tilde{a}_r + 1) M(\tilde{\mathbf{a}})$$

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$\mathbf{a} = (a_1, \dots, a_r)$ or $\tilde{\mathbf{a}} := (a_1 + 1, a_2, \dots, a_r, a_r - 1)$

corner c in f_1

$$(a_1 + 1) \lfloor a_r/2 \rfloor M(\mathbf{a}) = \lfloor \tilde{a}_1/2 \rfloor (\tilde{a}_r + 1) M(\tilde{\mathbf{a}})$$

Example: degree transfer in a bip. or quasibip. map

maps with r (rooted) faces $f_1, \ldots, f_r$, of degrees given by

$\mathbf{a} = (a_1, \ldots, a_r)$ or $\tilde{\mathbf{a}} := (a_1 + 1, a_2, \ldots, a_r, a_r - 1)$

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corner c' in f_r

Example: degree transfer in a bip. or quasibip. map

maps with r (rooted) faces $f_1, \ldots, f_r$, of degrees given by
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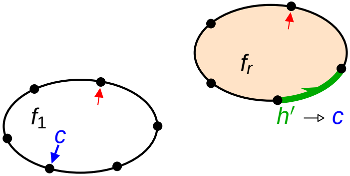
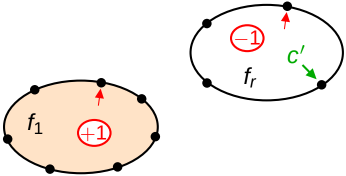
$$(a_1 + 1) \lfloor a_r/2 \rfloor M(\mathbf{a}) = \lfloor \tilde{a}_1/2 \rfloor (\tilde{a}_r + 1) M(\tilde{\mathbf{a}})$$

half-edge h' of f_r directed toward c

corner c' in f_r





Example: degree transfer in a bip. or quasibip. map

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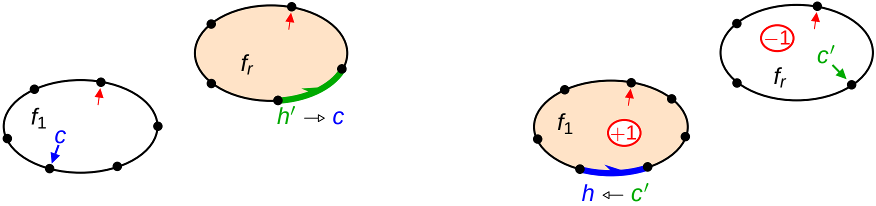
corner c in f_1

$$(a_1 + 1)\left\lfloor a_r/2 \right\rfloor M(\mathbf{a}) = \left\lfloor \tilde{a}_1/2 \right\rfloor (\tilde{a}_r + 1) M(\tilde{\mathbf{a}})$$

corner c' in f_r

half-edge h' of f_r directed toward c

half-edge h of f_1 directed away from c'



Example: degree transfer in a bip. or quasibip. map

maps with r (rooted) faces $f_1, \dots, f_r$, of degrees given by
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corner c in f_1

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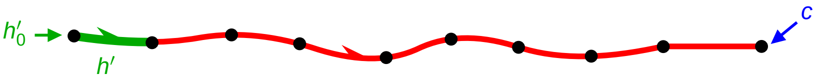

slit, slide, sew

Example: degree transfer in a bip. or quasibip. map

maps with r (rooted) faces $f_1, \ldots, f_r$, of degrees given by
$\mathbf{a} = (a_1, \ldots, a_r)$ or $\tilde{\mathbf{a}} := (a_1 + 1, a_2, \ldots, a_r, a_r - 1)$

corner c in f_1

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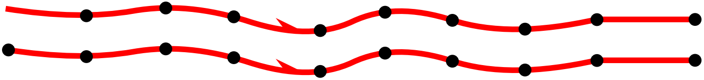

slit, slide, sew

Example: degree transfer in a bip. or quasibip. map

maps with r (rooted) faces $f_1, \ldots, f_r$, of degrees given by
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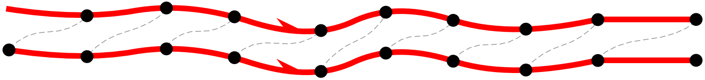
corner c in f_1

$$(a_1+1)\left\lfloor a_r/2 \right\rfloor M(\mathbf{a}) = \left\lfloor \tilde{a}_1/2 \right\rfloor (\tilde{a}_r+1) M(\tilde{\mathbf{a}})$$

corner c' in f_r

half-edge h' of f_r directed toward c

half-edge h of f_1 directed away from c'



slit, slide, sew

Example: degree transfer in a bip. or quasibip. map

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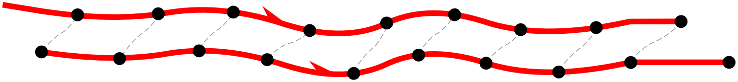

slit, slide, sew

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maps with r (rooted) faces $f_1, \ldots, f_r$, of degrees given by
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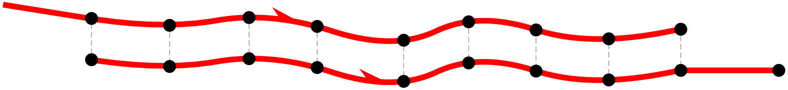

slit, slide, sew

Example: degree transfer in a bip. or quasibip. map

maps with r (rooted) faces $f_1, \dots, f_r$, of degrees given by
$\mathbf{a} = (a_1, \dots, a_r)$ or $\tilde{\mathbf{a}} := (a_1 + 1, a_2, \dots, a_r, a_r - 1)$

corner c in f_1

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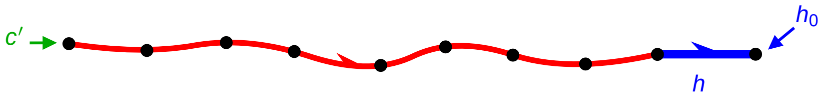
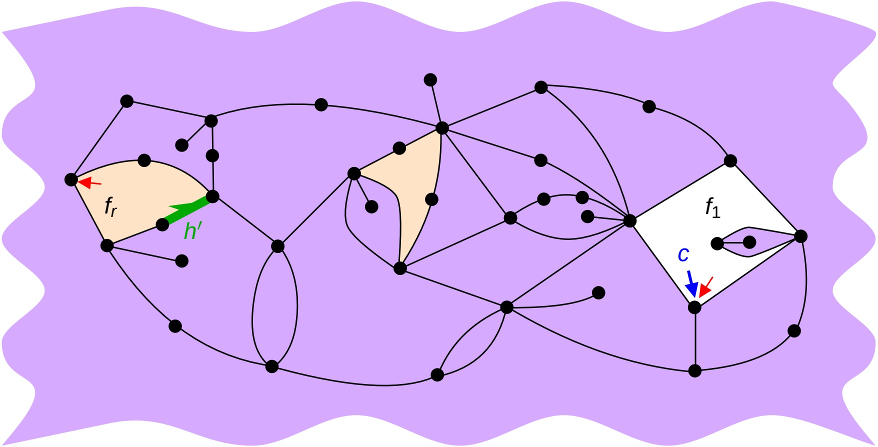
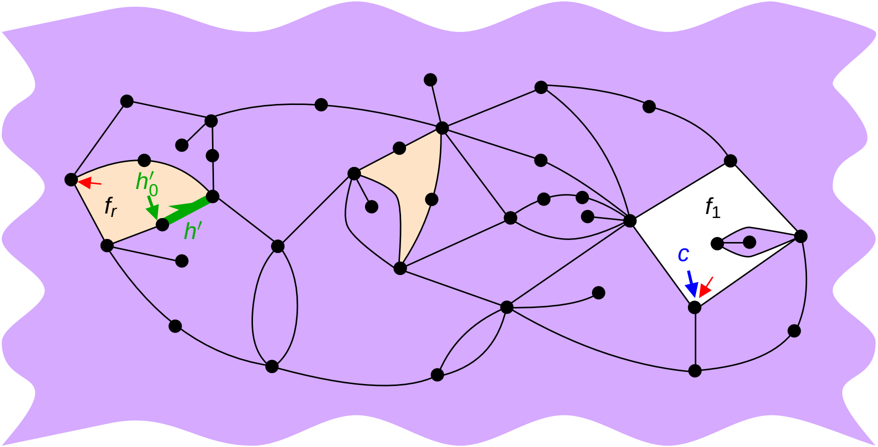


slit, slide, sew

Example: degree transfer in a bip. or quasibip. map

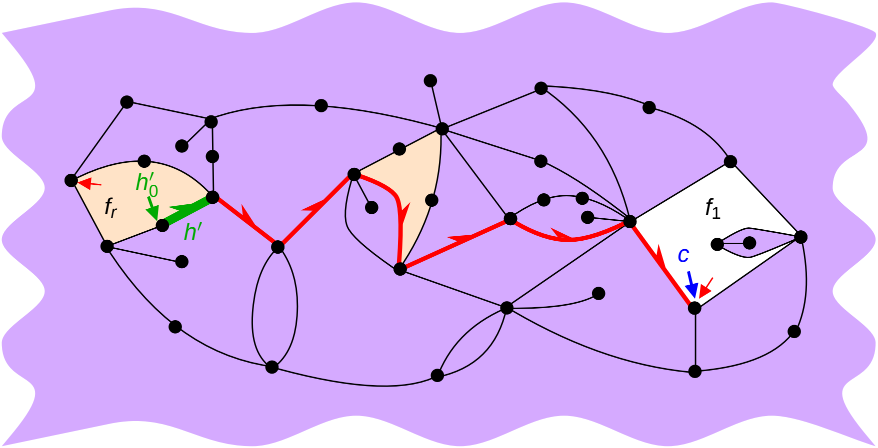


Example: degree transfer in a bip. or quasibip. map



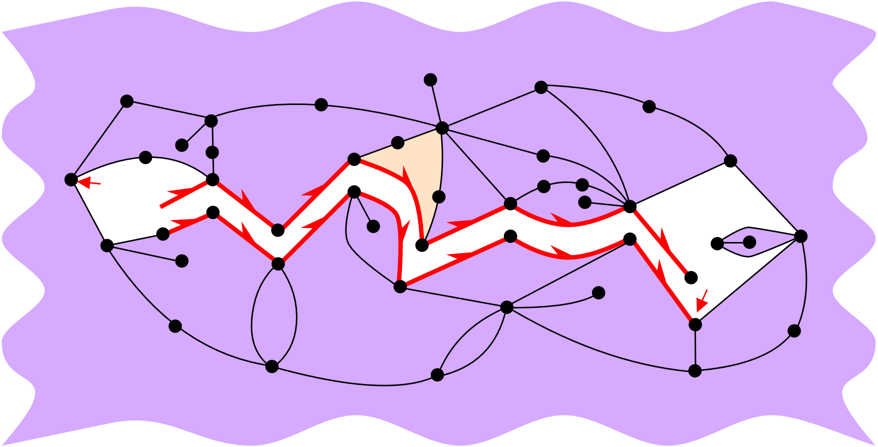
Consider the corner h'_0 delimited by h' and its predecessor in f_r.

Example: degree transfer in a bip. or quasibip. map



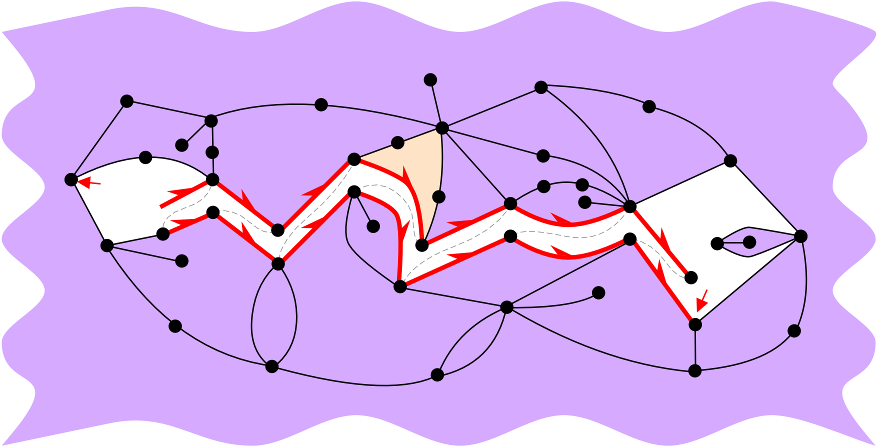
Consider the leftmost geodesic from h'_0 toward c.

Example: degree transfer in a bip. or quasibip. map



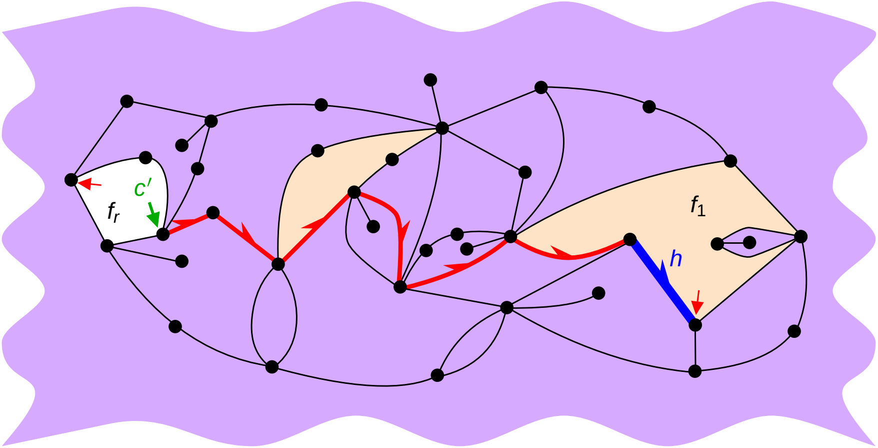
Slit!

Example: degree transfer in a bip. or quasibip. map



Slide!

Example: degree transfer in a bip. or quasibip. map



Sew! And mark h and c'.

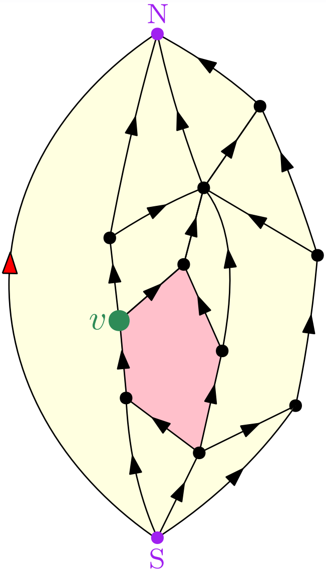
Back to bipolar oriented maps

$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right)(k+\ell-2)\, B_{k-1,\ell,j+1}$$

$\approx$ edge

internal vertex

Back to bipolar oriented maps



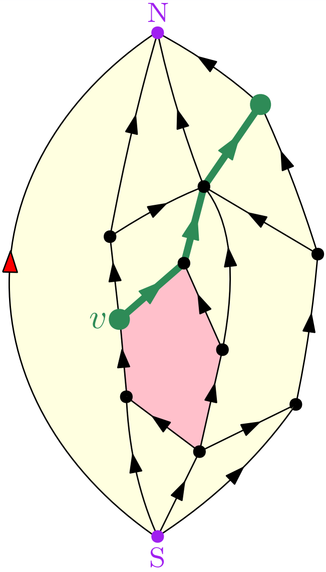
$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right) \left(k + \ell - 2\right) B_{k-1,\ell,j+1}$$

internal vertex

$\approx$ edge

Consider a b.o.m. with a marked internal vertex v.

Back to bipolar oriented maps



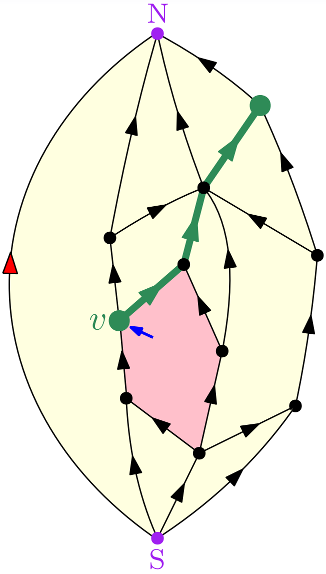
$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right)(k+\ell-2)\, B_{k-1,\ell,j+1}$$

internal vertex

$\approx$ edge

Take the rightmost path from v to N.

Back to bipolar oriented maps



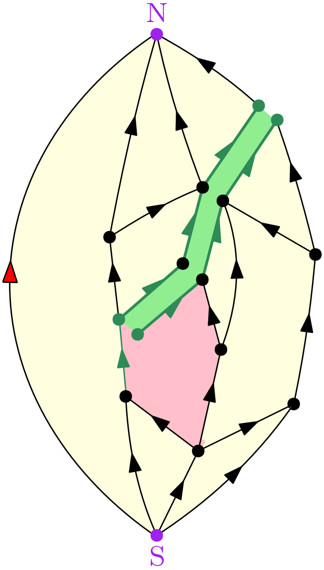
$$k\,B_{k,\ell,j} = \left(1-\frac{2}{j+1}\right)(k+\ell-2)\,B_{k-1,\ell,j+1}$$

internal vertex

$\approx$ edge

Enter from the face at the right of v and exit through the external face.

Back to bipolar oriented maps



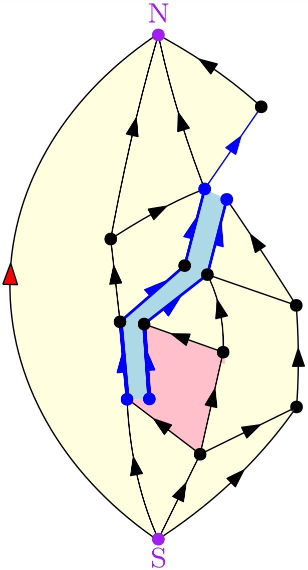
$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right)(k+\ell-2)\, B_{k-1,\ell,j+1}$$

$\approx$ edge

internal vertex

Slit!

Back to bipolar oriented maps



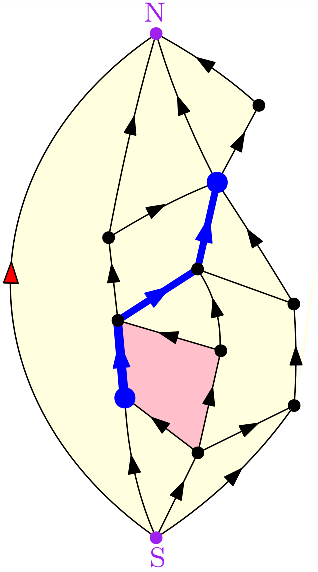
$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right)(k+\ell-2)\, B_{k-1,\ell,j+1}$$

internal vertex

$\approx$ edge

Slide!

Back to bipolar oriented maps



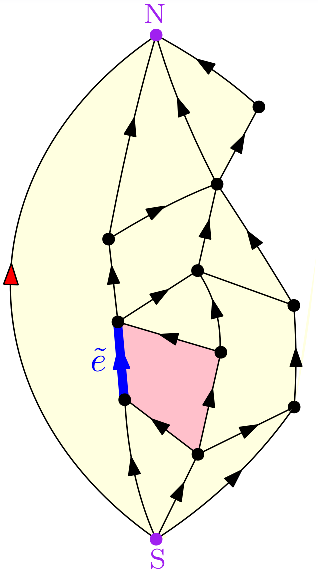
$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right)(k+\ell-2)\, B_{k-1,\ell,j+1}$$

$\approx$ edge

internal vertex

Sew!

Back to bipolar oriented maps



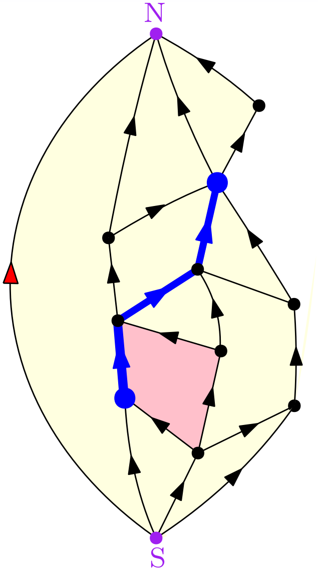
$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right)(k+\ell-2)\, B_{k-1,\ell,j+1}$$

internal vertex

≈ edge

Mark $\tilde{e}$, the first edge of the resulting path.

Back to bipolar oriented maps



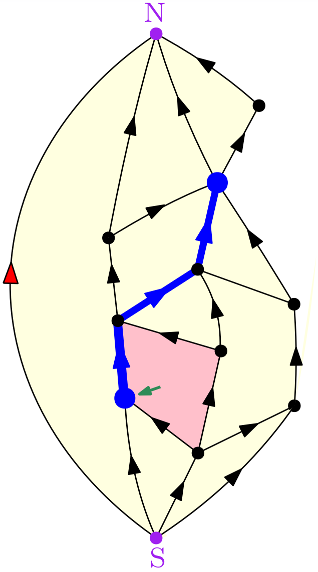
$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right)(k+\ell-2)\, B_{k-1,\ell,j+1}$$

internal vertex

≈ edge

Conversely, take the rightmost path from $\tilde{e}$ to N.

Back to bipolar oriented maps



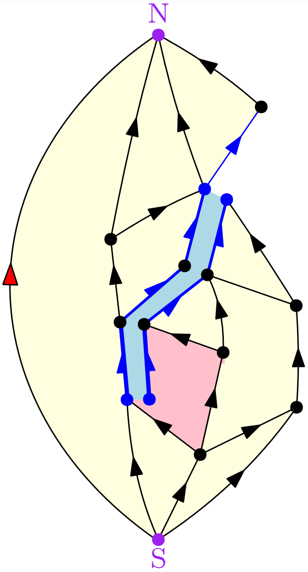
$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right) (k+\ell-2)\, B_{k-1,\ell,j+1}$$

internal vertex

$\approx$ edge

Enter from the right of the first vertex of the path, and exit through the external face.

Back to bipolar oriented maps



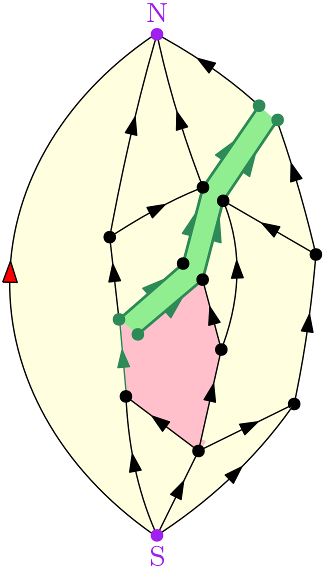
$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right)(k+\ell-2)\, B_{k-1,\ell,j+1}$$

internal vertex

$\approx$ edge

Slit!

Back to bipolar oriented maps



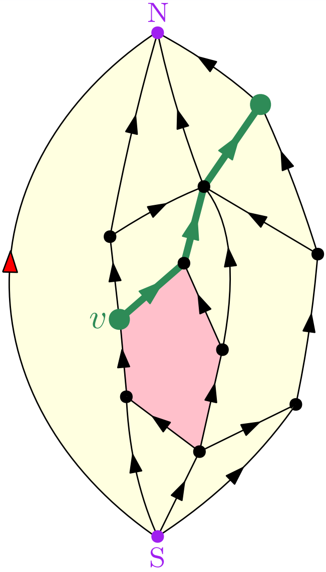
$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right)\left(k+\ell-2\right) B_{k-1,\ell,j+1}$$

internal vertex

$\approx$ edge

Slide!

Back to bipolar oriented maps



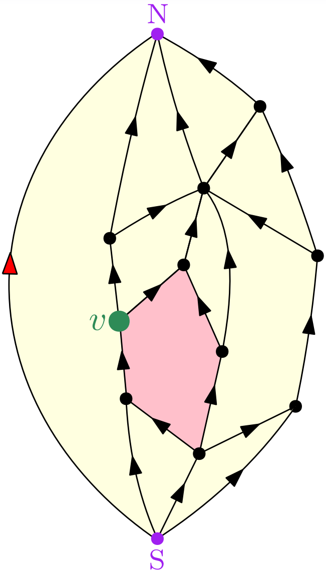
$$\boxed{k}\, B_{k,\ell,j} = \boxed{\left(1 - \frac{2}{j+1}\right)} \boxed{(k+\ell-2)}\, B_{k-1,\ell,j+1}$$

internal vertex

$\approx$ edge

Sew!

Back to bipolar oriented maps



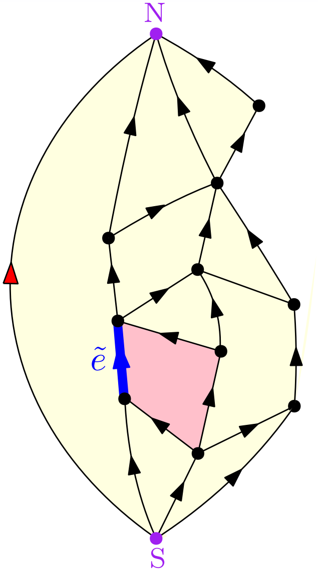
$$k\, B_{k,\ell,j} = \left(1-\frac{2}{j+1}\right)(k+\ell-2)\, B_{k-1,\ell,j+1}$$

internal vertex

$\approx$ edge

Mark v, the first vertex of the resulting path.

Back to bipolar oriented maps

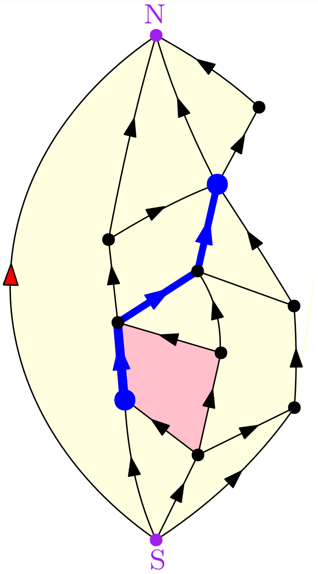


$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right)(k+\ell-2)\, B_{k-1,\ell,j+1}$$

internal vertex

(nonroot) edge with internal face to its right

Back to bipolar oriented maps



probability for the path to reach the boundary before N

$$k\, B_{k,\ell,j} = \left(1 - \frac{2}{j+1}\right)(k+\ell-2)\, B_{k-1,\ell,j+1}$$

internal vertex

(nonroot) edge with internal face to its right

It is a bijection between

bipolar oriented maps carrying a marked internal vertex

and

bipolar oriented maps carrying a marked edge

- having an internal face to its right
- and whose rightmost path to N reaches the boundary before N

Through the bijection from top to bottom:

- *the external degree increases by one;*
- *the number of internal vertices decreases by one;*
- *the number of internal faces and their right lengths are preserved;*
- *the left lengths are preserved, except that of the face at the right of the marked vertex, which decreases by one.*

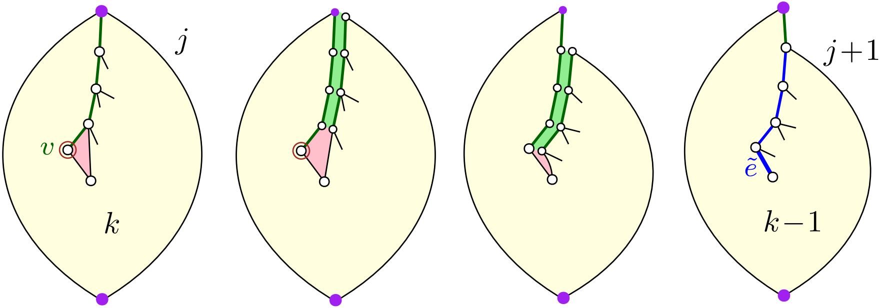
Specializations

Schnyder woods. Ok since right lengths are preserved.

Specializations

Schnyder woods. Ok since right lengths are preserved.

Quasi-triangulations. The face at the right of the distinguished vertex becomes of degree 2. We “squeeze” it into the marked edge.



The marked edge no longer necessarily has an internal face to its right.

Boundary-reaching ratio

b.o.m. with a marked boundary-reaching internal edge

$$\frac{|\mathcal{B}^{\partial\times}_{k,\ell,j}|}{|\mathcal{B}^{\times}_{k,\ell,j}|}$$

b.o.m. with a marked internal edge

k internal vertices
ℓ internal faces
external degree j

Boundary-reaching ratio

b.o.m. with a marked boundary-reaching[☝] internal edge

$$\frac{|\mathcal{B}^{\partial \times}_{k,\ell,j}|}{|\mathcal{B}^{\times}_{k,\ell,j}|}$$

b.o.m. with a marked internal edge

k internal vertices
ℓ internal faces
external degree j

[☝] whose rightmost path to N reaches the boundary before N

Boundary-reaching ratio

b.o.m. with a marked boundary-reaching internal edge

Remains to show that the ratio $\dfrac{|\mathcal{B}^{\partial \times}_{k,\ell,j}|}{|\mathcal{B}^{\times}_{k,\ell,j}|} = 1 - \dfrac{2}{j}$.

k internal vertices
ℓ internal faces
external degree j

b.o.m. with a marked internal edge

---

whose rightmost path to N reaches the boundary before N

Boundary-reaching ratio

b.o.m. with a marked boundary-reaching[*] ~~internal~~ edge

Remains to show that the ratio $\frac{|\mathcal{B}^{\partial\nearrow}_{k,\ell,j}|}{|\mathcal{B}^{\nearrow}_{k,\ell,j}|} = 1 - \frac{2}{j}$.

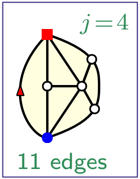
k internal vertices
ℓ internal faces
external degree j

b.o.m. with a marked ~~internal~~ edge

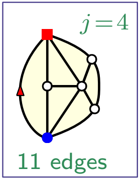
[*] whose rightmost path to N reaches the boundary before N

Is $2/j$ the proportion of non boundary-reaching edges?

for a fixed bipolar oriented map?

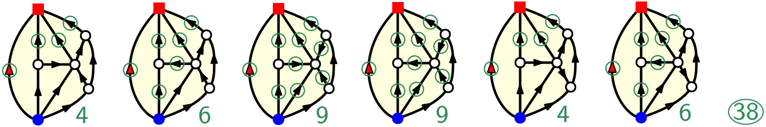


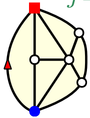
Is $2/j$ the proportion of non boundary-reaching edges?



for a fixed rooted map
(considering all its possible orientation)?

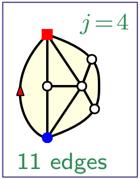
Is $2/j$ the proportion of non boundary-reaching edges?





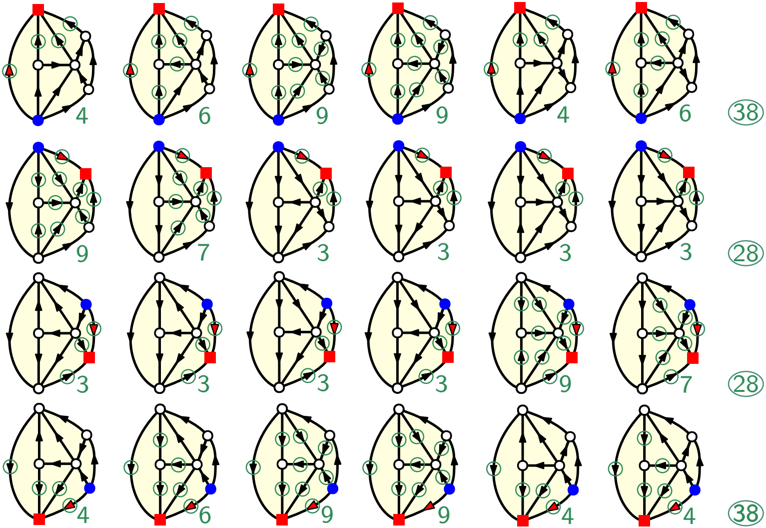
$$\neq \frac{2}{4}(11 \times 6) = 33$$

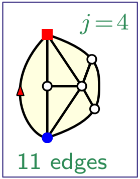
Is $2/j$ the proportion of non boundary-reaching edges?



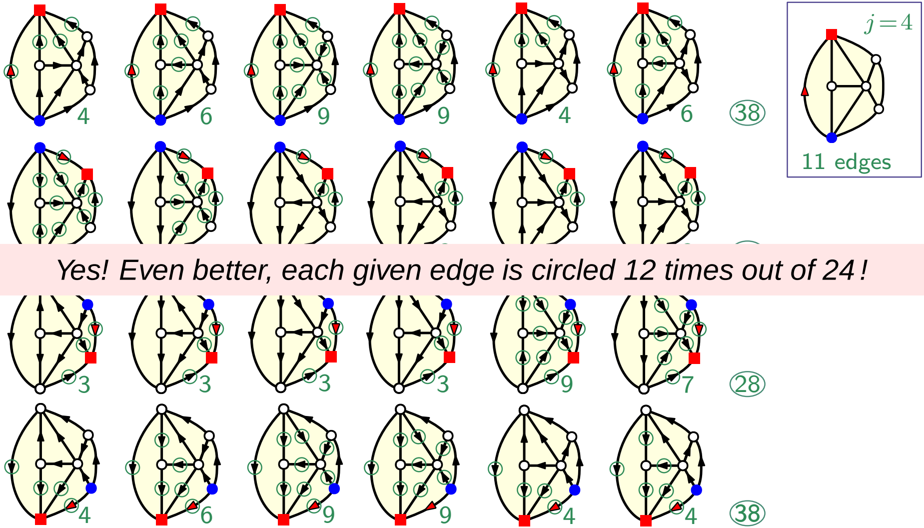
for a fixed map up to rerooting
on the boundary?

Is $2/j$ the proportion of non boundary-reaching edges?





Is $2/j$ the proportion of non boundary-reaching edges?

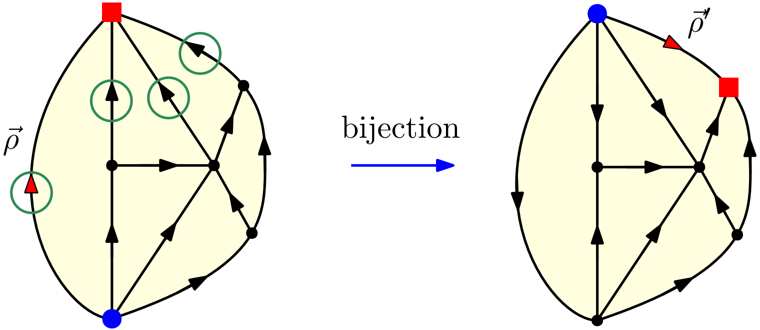


Yes! Even better, each given edge is circled 12 times out of 24 !

Rerooting operator

Rerooting bijection [de Fraysseix et al. '95]

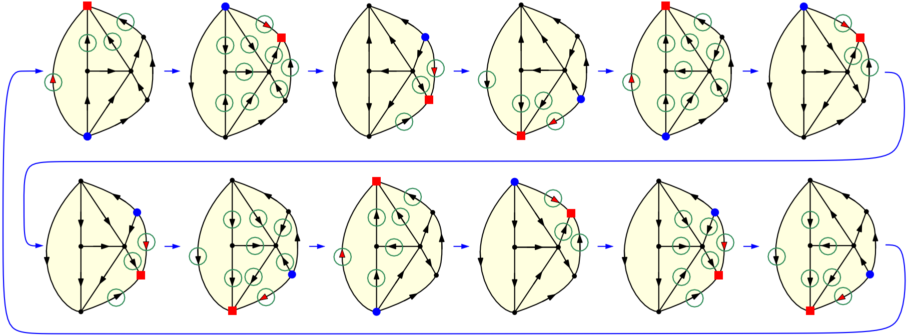
- *Take as new root the next in clockwise order around the boundary.*
- *Return non boundary-reaching edges.*



The number of bipolar orientations does not depend on the root choice!

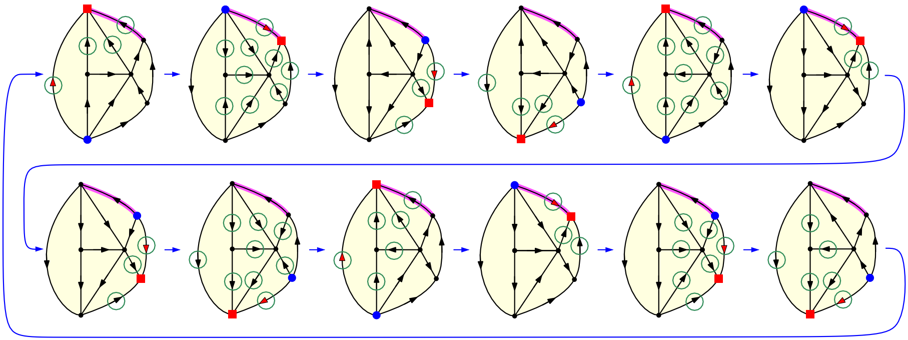
Orbit property

Along an orbit, any edge is not boundary-reaching $2/j$ of the time.



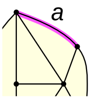
Orbit property

Along an orbit, any edge is not boundary-reaching $2/j$ of the time.



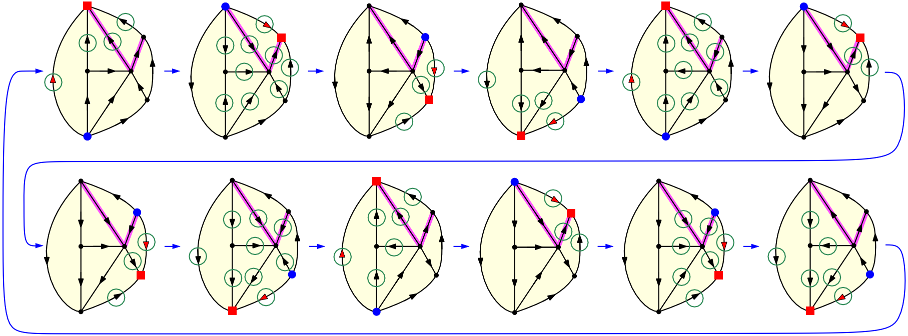
easy for outer edges

$$a\colon\ \text{✗✗}\underbrace{\checkmark\cdots\checkmark}_{j-2}\text{✗✗}\underbrace{\checkmark\cdots\checkmark}_{j-2}$$



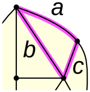
Orbit property

Along an orbit, any edge is not boundary-reaching $2/j$ of the time.



easy for outer edges

a: ✗✗$\underbrace{\checkmark\cdots\checkmark}_{j-2}$✗✗$\underbrace{\checkmark\cdots\checkmark}_{j-2}$



propagation around vertices

b: ✗✗✓✓✗✓✓✗✗✓✗✓

c: ✓✗✓✓✗✗✓✗✓✗✗✓