Brownian surfaces

Jérémie BETTINELLI

June 5, 2023

What is a map?

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Village de la Putain
Temple du Rock
Citadelle des impôts
Grotte satanique
La cité de la Mort
La cité perdue
Taste of ink Tattoo
Île du naufragé
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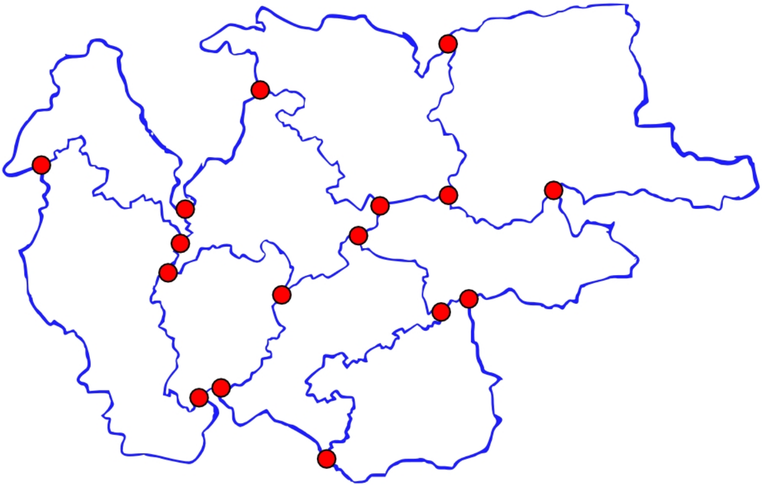

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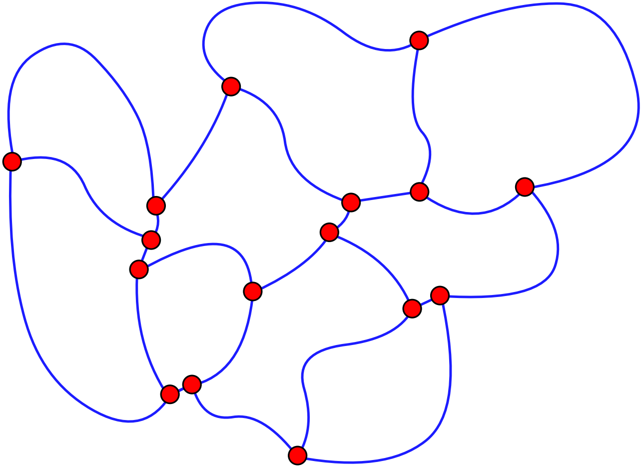

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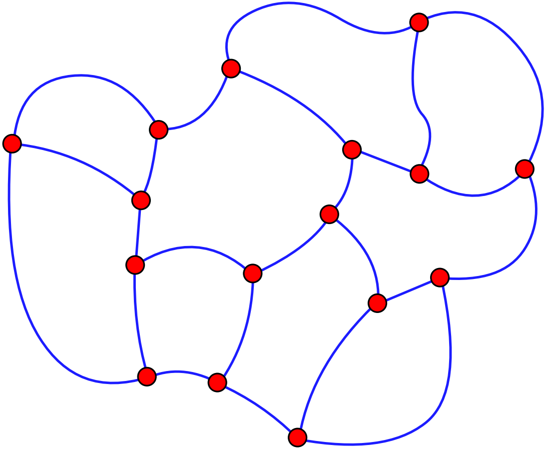
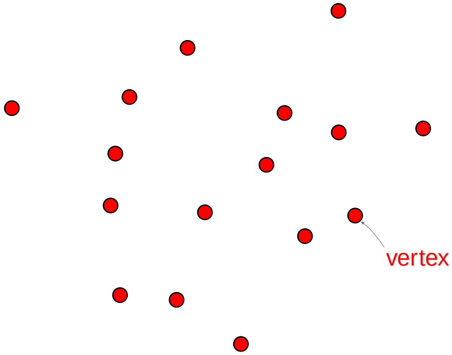


Plane map, formally



- We have **vertices**

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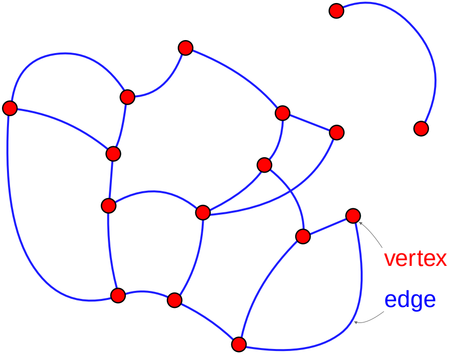

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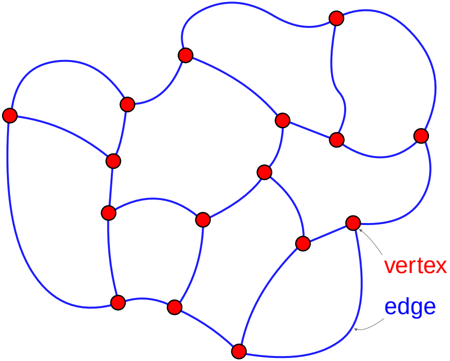

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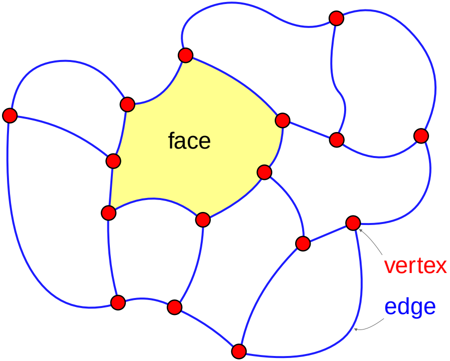

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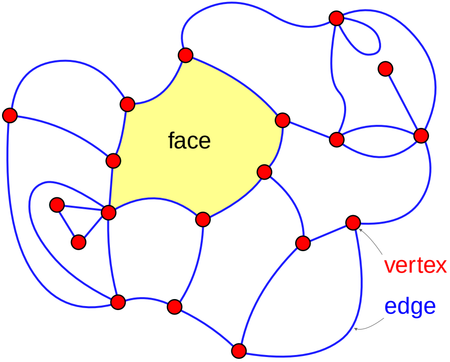

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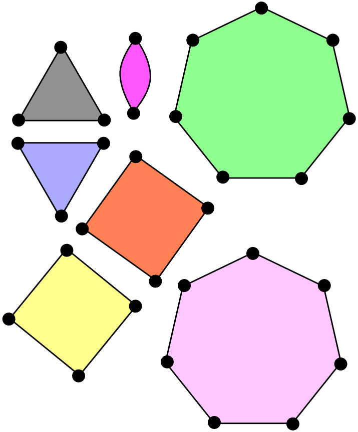
- Multiple edges and loops are allowed.

Other point of view: gluing of polygons



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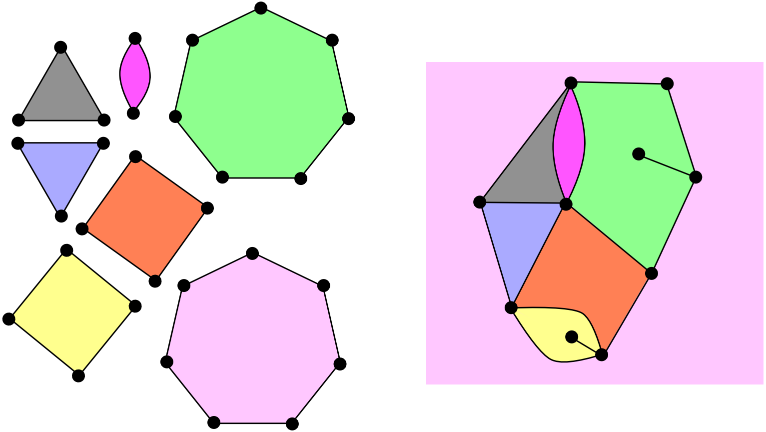

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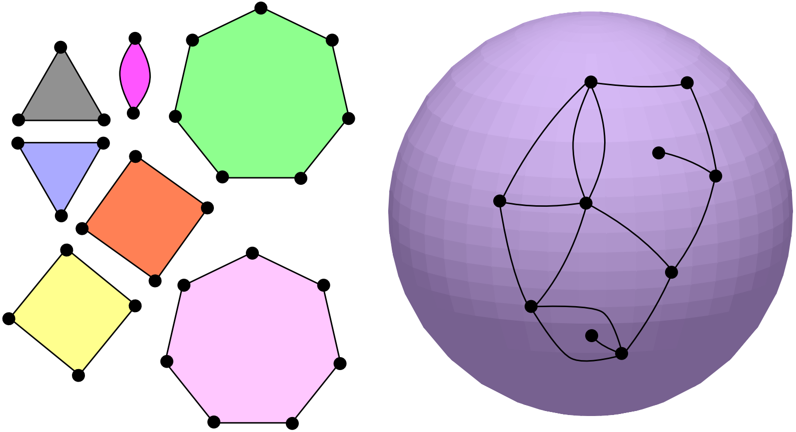
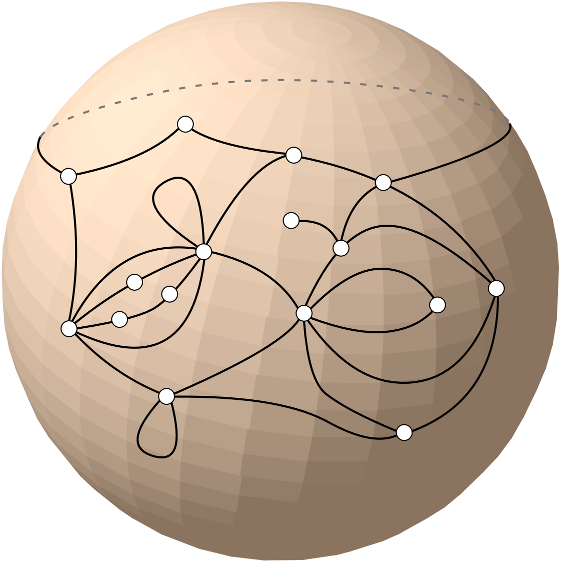


Root = distinguished corner



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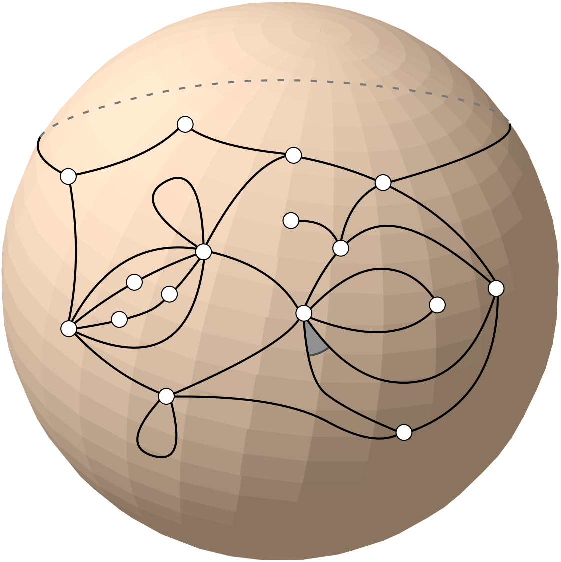
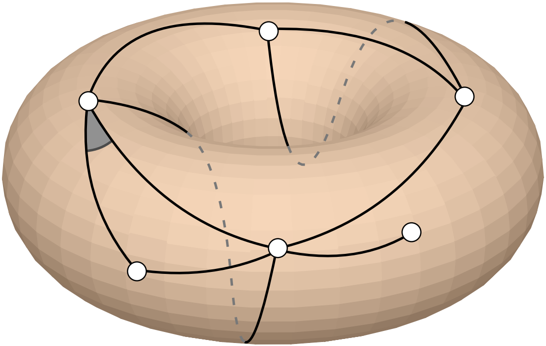


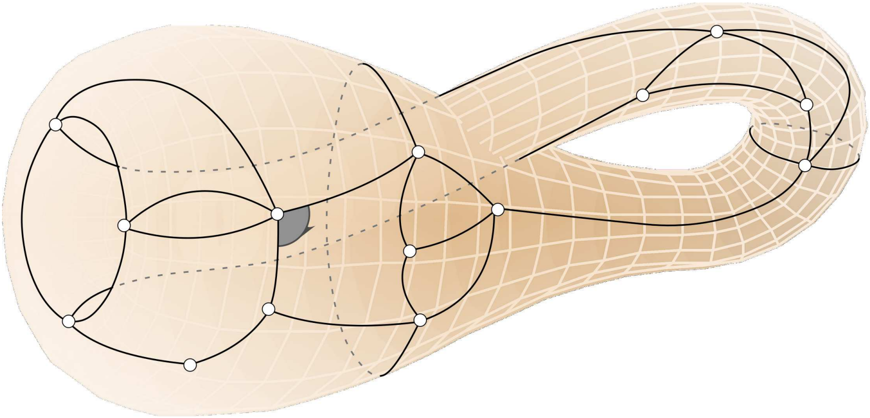
Genus g maps



genus g map: graph embedded in the surface of genus g, in such a way that the faces are homeomorphic to disks

maps are defined up to direct homeomorphism of the underlying surface

Nonorientable maps



root: distinguished corner given with a local orientation

maps are defined up to homeomorphism of the underlying surface

Why study maps?

- Very rich combinatorics.

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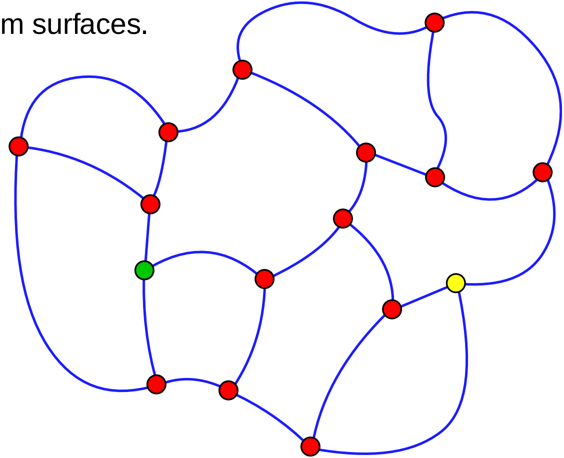

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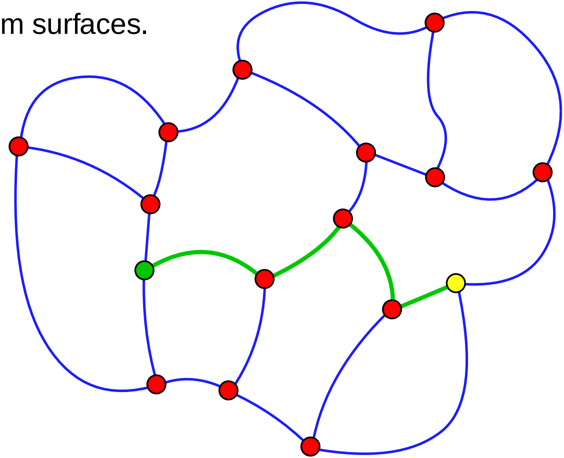

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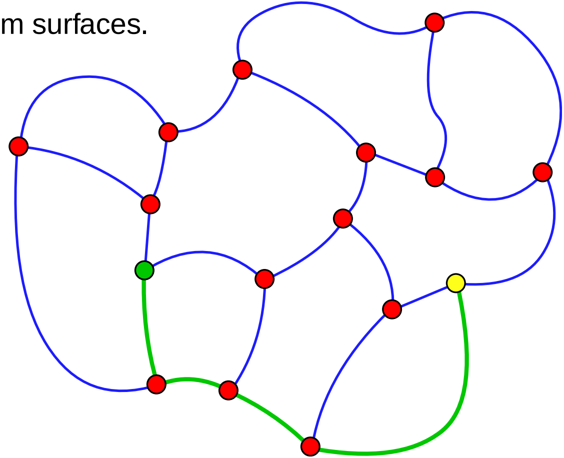

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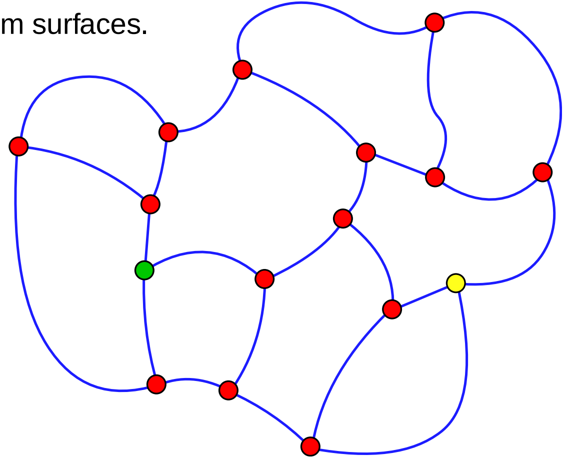

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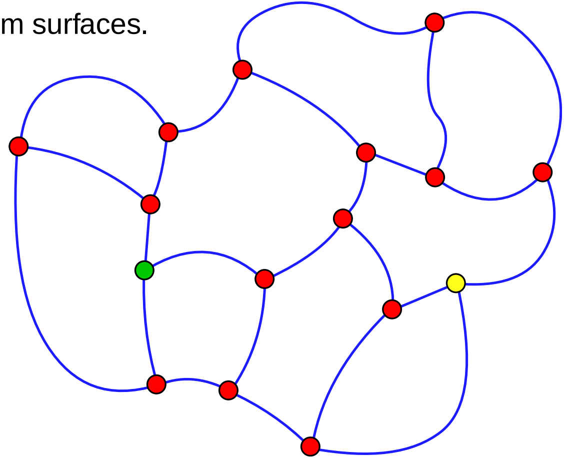
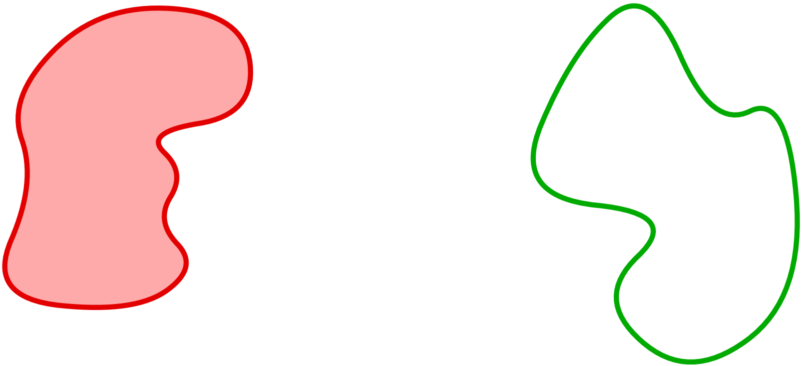


- They are beautiful!
- They share deep links with trees.

Gromov–Hausdorff topology



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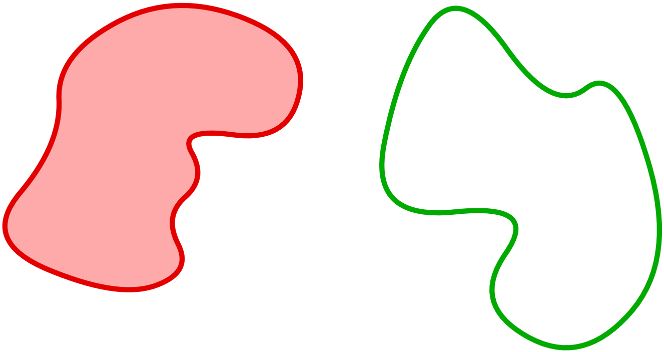

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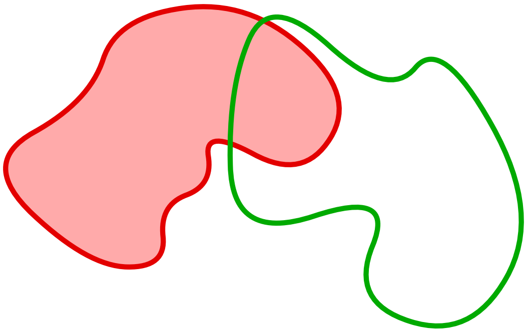

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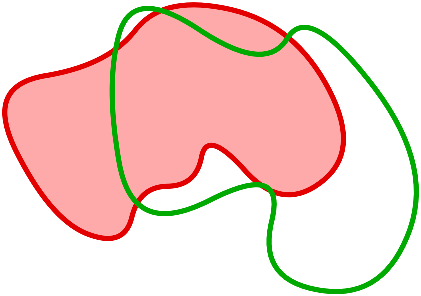

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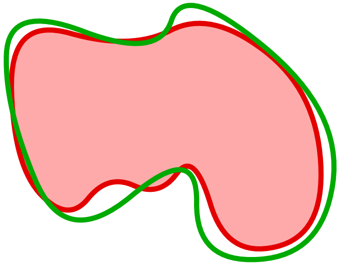

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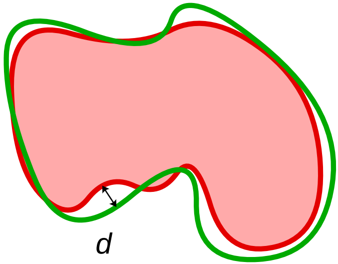

The Brownian sphere

- $a\,\mathbf{m}$: finite metric space obtained by endowing the vertex-set of $\mathbf{m}$ with a times the graph metric (each edge has length a).

Theorem (Le Gall '11, Miermont '11)

Let $\mathbf{q}_n$ be a uniform plane quadrangulation with n faces. The sequence $\left((8n/9)^{-1/4}\mathbf{q}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space called the Brownian sphere.

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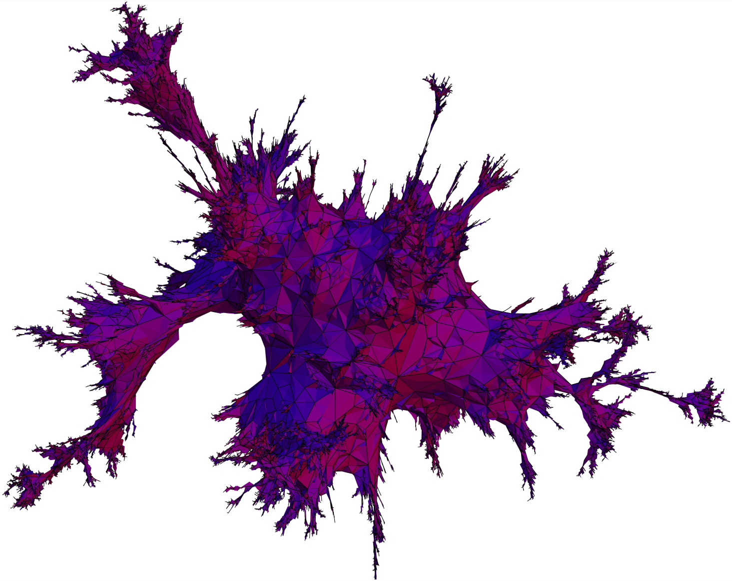
Definition (Convergence for the Gromov–Hausdorff topology)

A sequence $(\mathcal{X}_n)$ of compact metric spaces **converges in the sense of the Gromov–Hausdorff topology** toward a metric space $\mathcal{X}$ if there exist isometric embeddings $\varphi_n : \mathcal{X}_n \to \mathcal{Z}$ and $\varphi : \mathcal{X} \to \mathcal{Z}$ into a common metric space $\mathcal{Z}$ such that $\varphi_n(\mathcal{X}_n)$ converges toward $\varphi(\mathcal{X})$ in the sense of the Hausdorff topology.

Uniform plane quadrangulation with 50 000 faces



Earlier results

- [Chassaing–Schaeffer '04]
  - the scaling factor is $n^{1/4}$
  - scaling limit of functionals of random uniform quadrangulations (radius, profile)
- [Marckert–Mokkadem '06]
  - introduction of the Brownian sphere (called Brownian map)
- [Le Gall '07]
  - the sequence of rescaled quadrangulations is relatively compact
  - any subsequential limit has the topology of the Brownian sphere
  - any subsequential limit has Hausdorff dimension 4
- [Le Gall–Paulin '08], [Miermont '08]
  - the topology of any subsequential limit is that of the two-sphere
- [Bouttier–Guitter '08]
  - limiting joint distribution between three uniformly chosen vertices

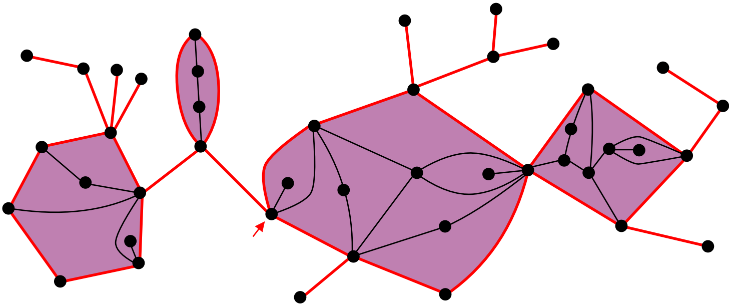
Universality of the Brownian sphere

Many other natural models of plane maps converge to the Brownian sphere (up to a model-dependent scale constant):

$$c\, n^{-1/4}\, \mathbf{m}_n \xrightarrow[n\to\infty]{} \text{Brownian sphere.}$$

- [Le Gall '11] uniform p-angulations for $p \in \{3, 4, 6, 8, 10, \ldots\}$ and Boltzmann bipartite maps with fixed number of vertices
- [Beltran and Le Gall '12] quadrangulations with no pendant edges
- [Addario-Berry–Albenque '13] simple triangulations and simple quadrangulations
- [B.–Jacob–Miermont '14] maps with fixed number of edges
- [Abraham '14] bipartite maps with fixed number of edges
- [Marzouk '17] bipartite maps with prescribed degree sequence
- [Curien–Le Gall '19] random length plane triangulations
- [Addario-Berry–Albenque '20] p-angulations for odd $p \geq 5$
- [Marzouk '20] planes bipartites maps with prescribed degrees

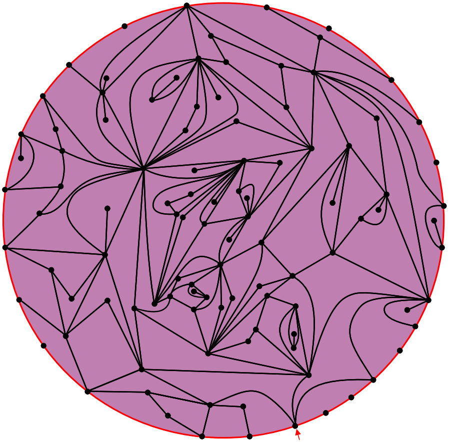
Plane quadrangulation with a boundary



plane map whose faces have degree 4, except possibly the root face

the boundary is not necessarily a simple curve

Plane quadrangulation with a simple boundary



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Brownian disks

- $\mathbf{q}_{n,p}$ uniform among quadrangulations with a boundary having area n and perimeter p
- $\ell_n/\sqrt{2n} \to L \in (0,\infty)$

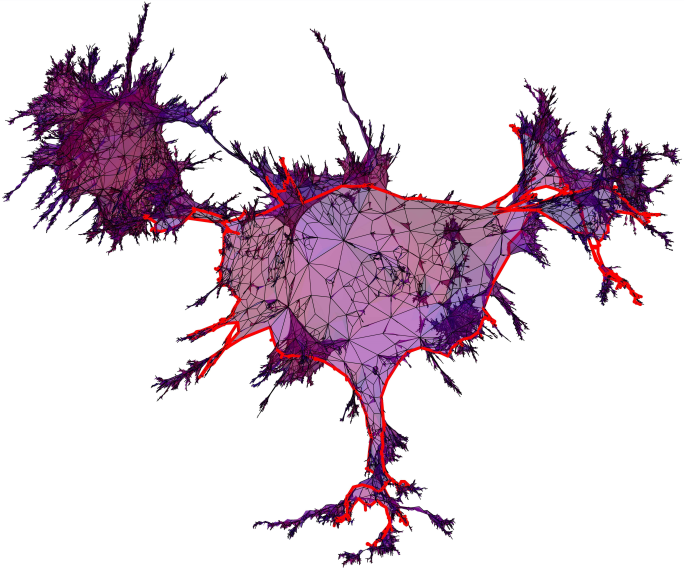
Theorem (B.–Miermont '15)

The sequence $\left((8n/9)^{-1/4}\mathbf{q}_{n,2\ell_n}\right)_{n\geq 1}$ *converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space* $\mathbf{BD}_L$ *called the Brownian disk of perimeter* L.

Theorem (B. '11)

Let $L > 0$ *be fixed. Almost surely, the space* $\mathbf{BD}_L$ *is homeomorphic to the closed unit disk of* $\mathbb{R}^2$. *Moreover, almost surely, the Hausdorff dimension of* $\mathbf{BD}_L$ *is* 4, *while that of its boundary* $\partial\mathbf{BD}_L$ *is* 2.

40 000 faces and boundary length 1 000



Universality

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Theorem (B.–Curien–Fredes–Sepúlveda '21)

The sequence $\left((8n/9)^{-1/4}\tilde{\mathbf{q}}_{n,2\ell_n}\right)_{n\geq 1}$ *converges weakly in the sense of the Gromov–Hausdorff topology toward* $\mathbf{BD}_{3L}$*, the Brownian disk of perimeter* $3L$*.*

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- [B.–Miermont '15] $2p$-ang., uniform bip. maps, bip. Boltzmann maps
- [Gwynne–Miller '19] Boltzmann quad. with a simple boundary
- [Albenque–Holden–Sun '20] Boltzmann tri. with a simple boundary

Degenerate regimes

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Theorem (B. '11) $\ell_n/\sqrt{2n} \to 0$

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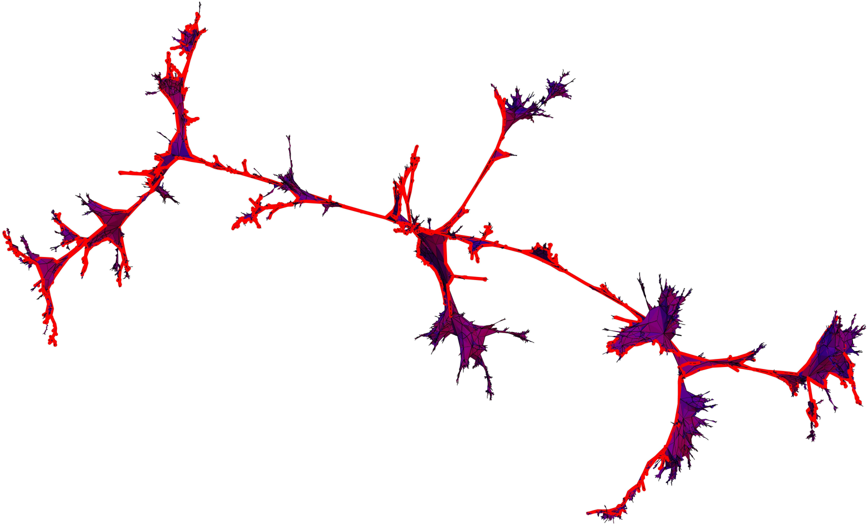
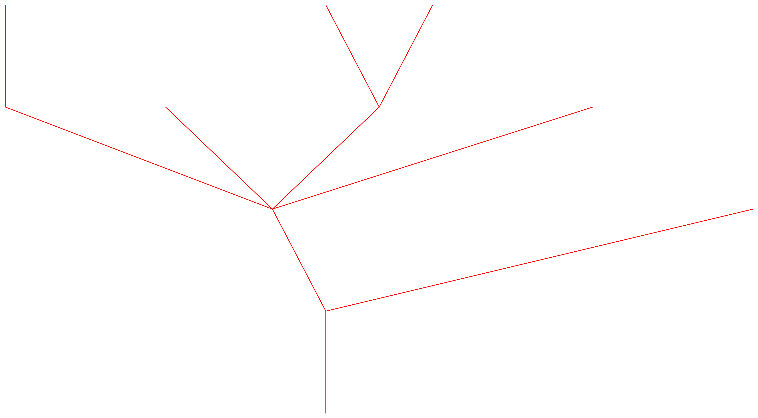
- [Bouttier–Guitter '09] computation of the two-point function
- [Marzouk '20] bipartite maps with prescribed degrees

10 000 faces and boundary length 2 000



The Continuum Random Tree



tree

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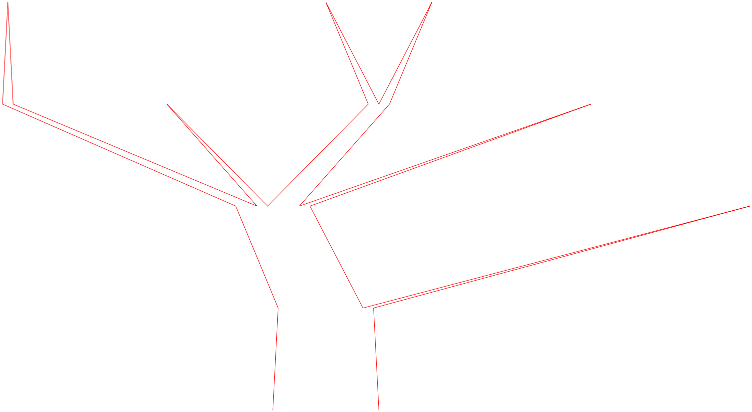

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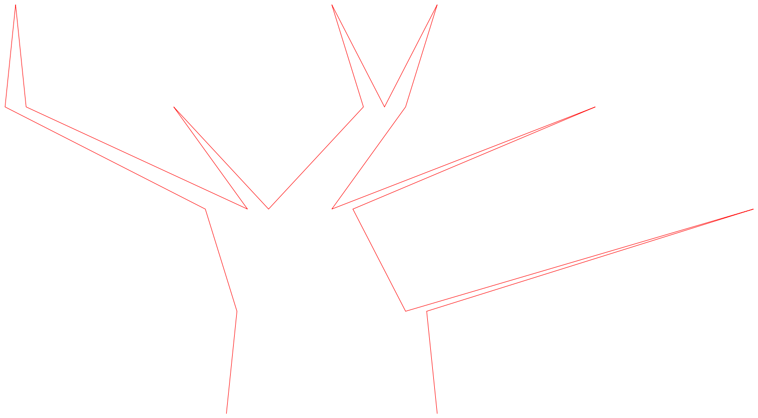

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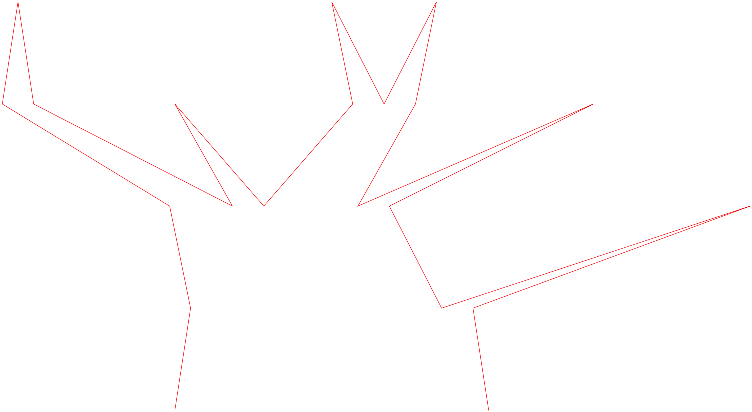

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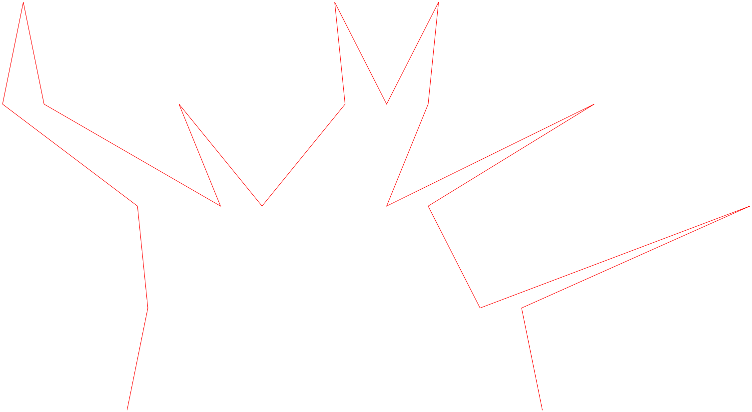

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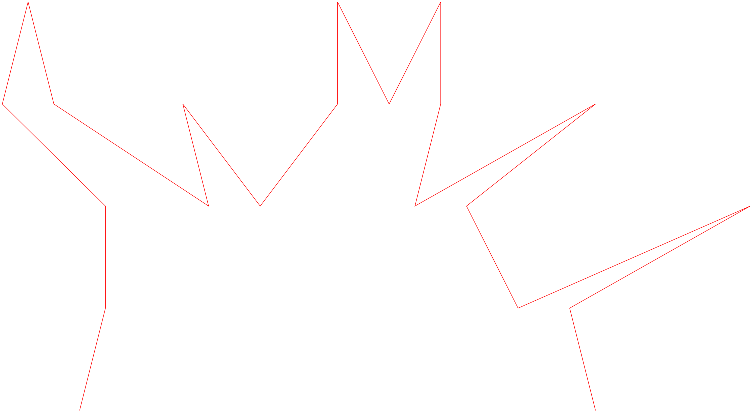

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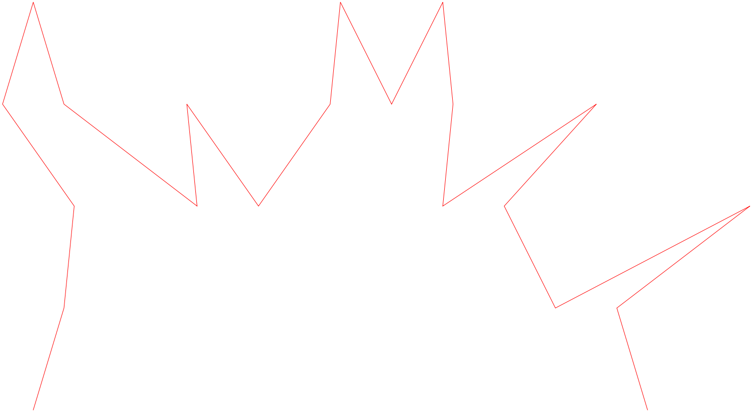

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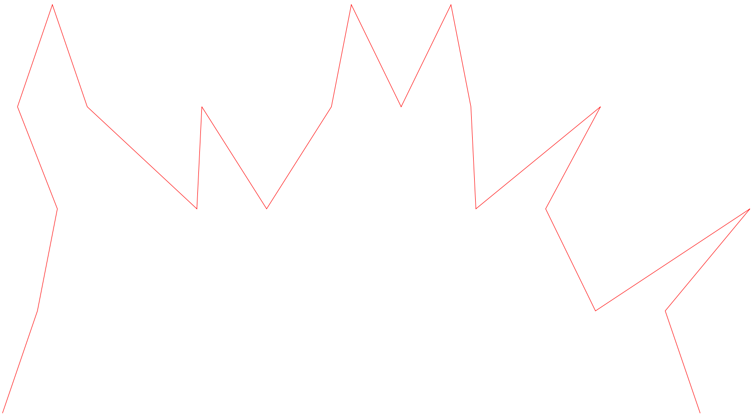

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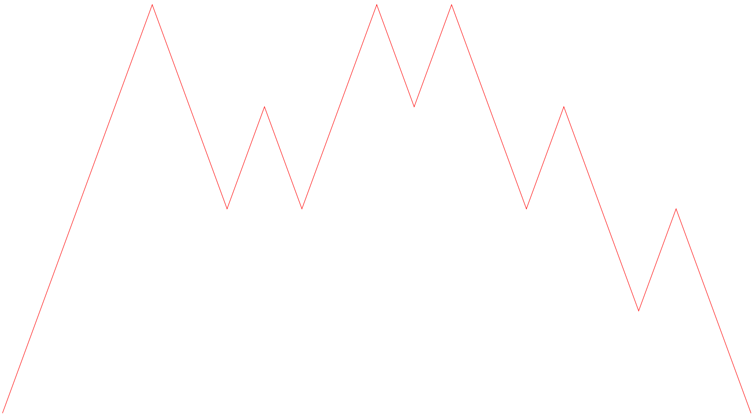

Dyck path

The Continuum Random Tree

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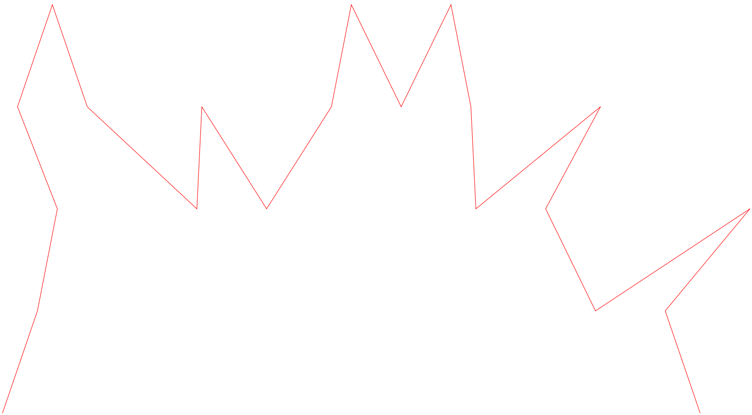

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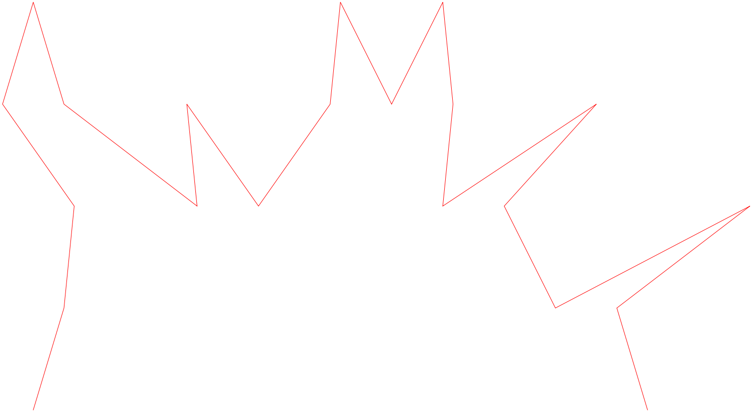

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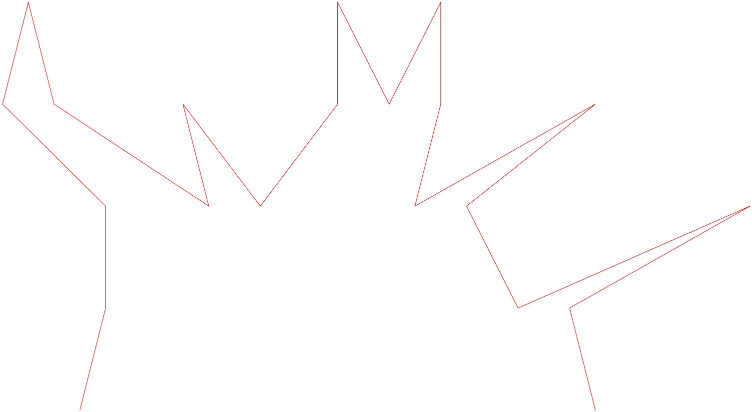

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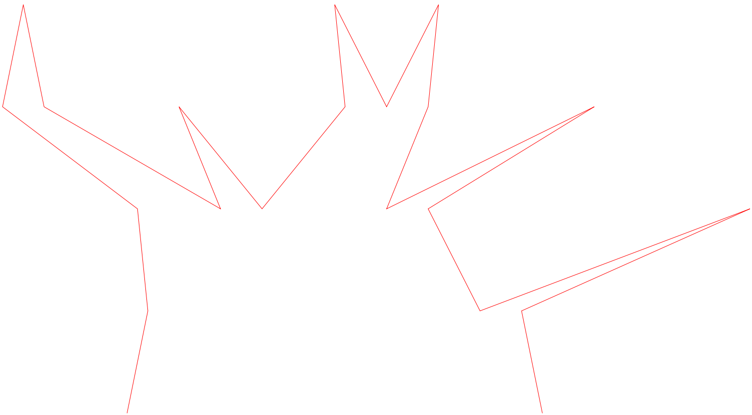

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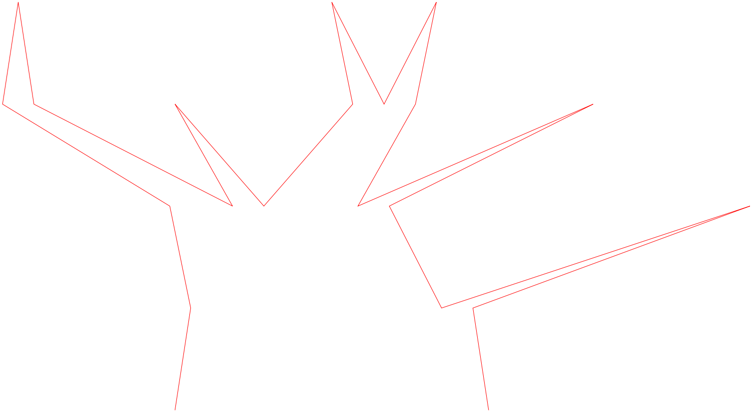

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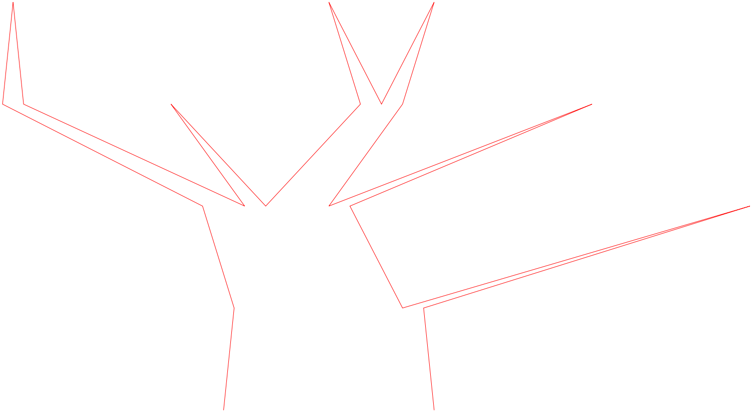

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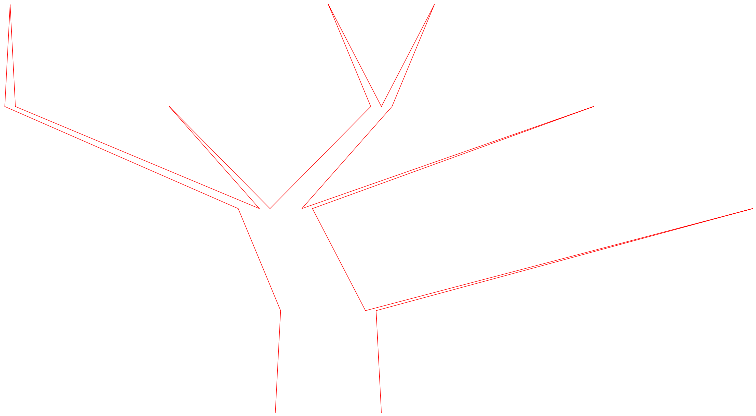

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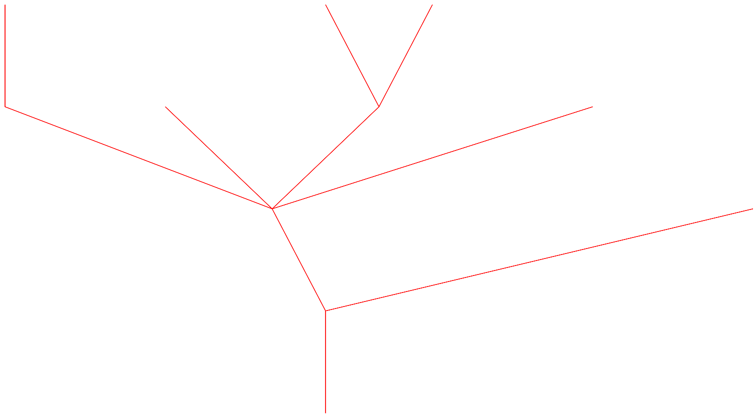
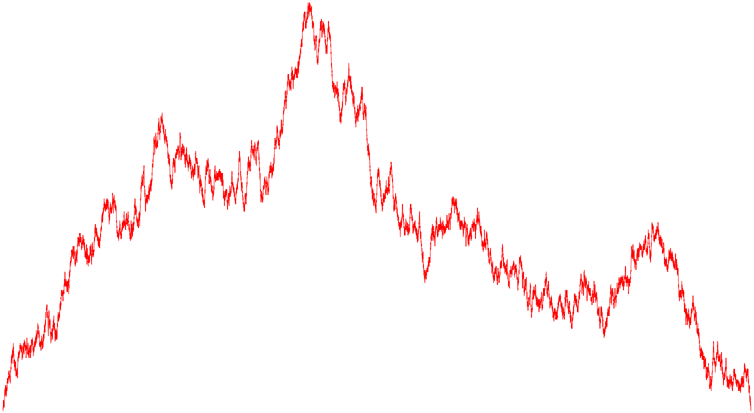


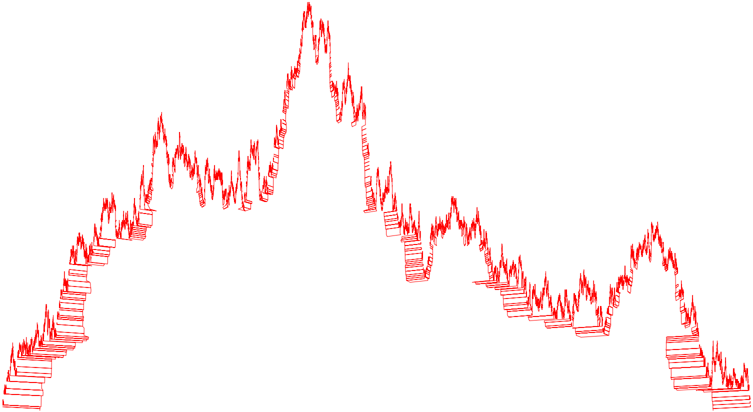
tree

The Continuum Random Tree

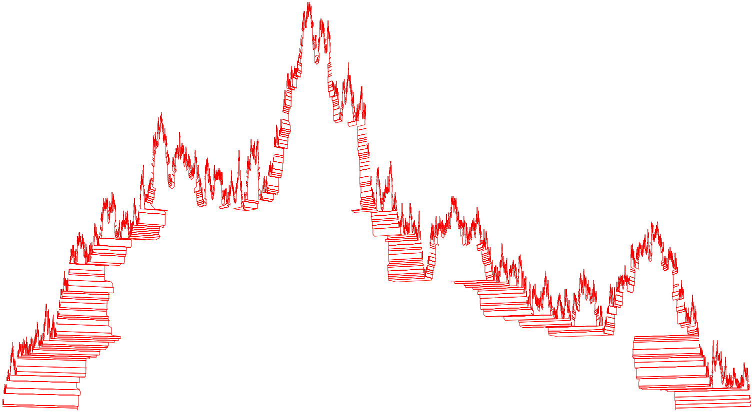


Brownian excursion: Brownian motion on $[0,1]$ conditioned to stay positive on $(0,1)$ and be back at 0 at time 1

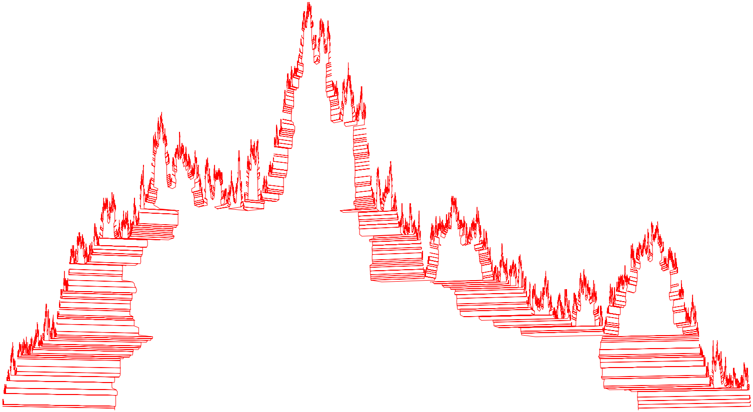
The Continuum Random Tree



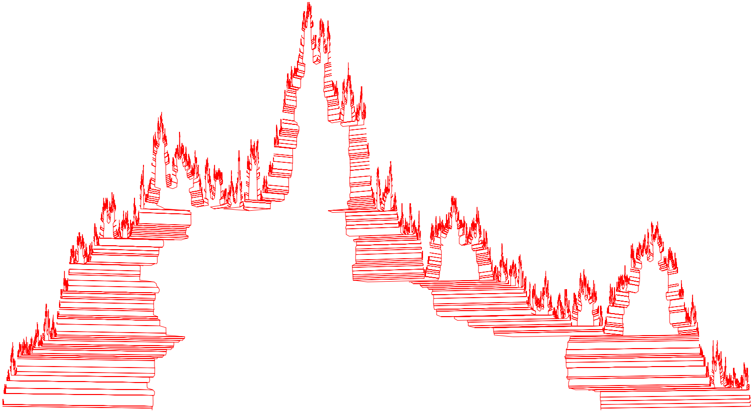
The Continuum Random Tree



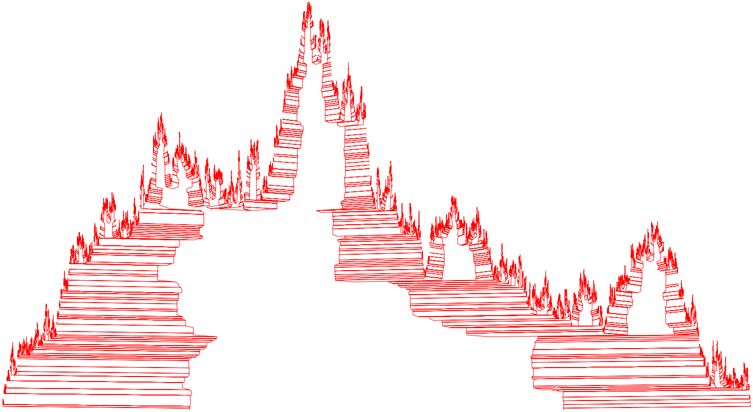
The Continuum Random Tree



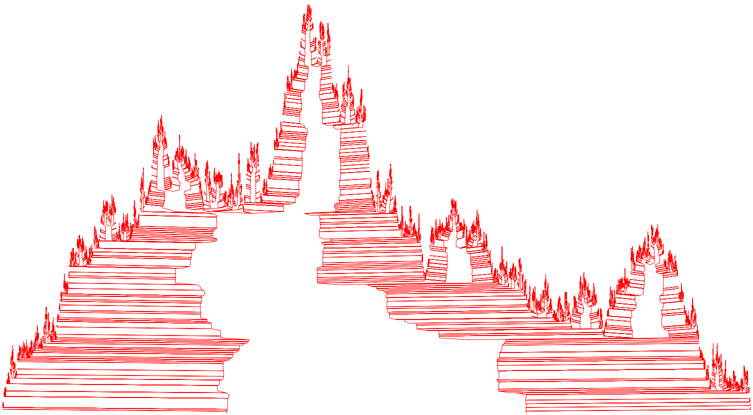
The Continuum Random Tree



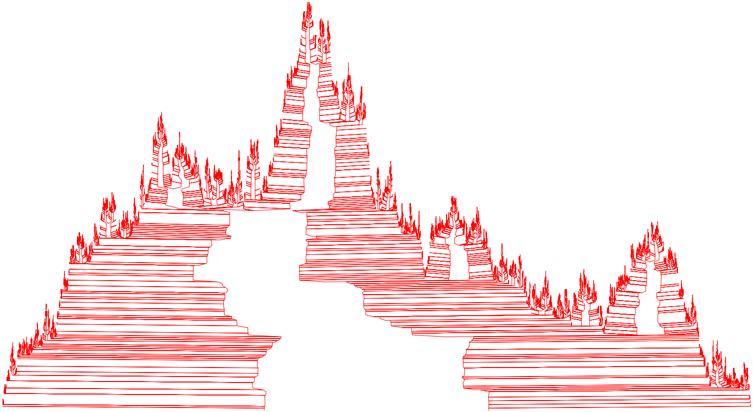
The Continuum Random Tree



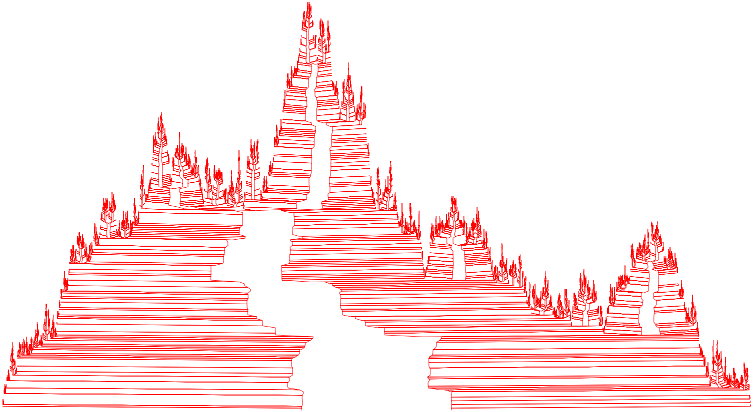
The Continuum Random Tree



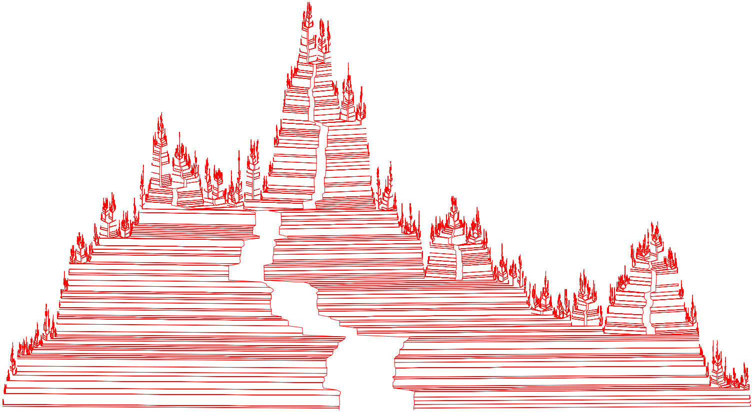
The Continuum Random Tree



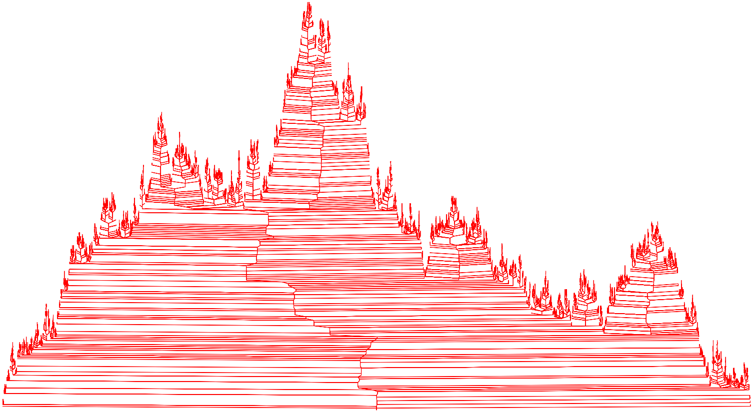
The Continuum Random Tree



The Continuum Random Tree

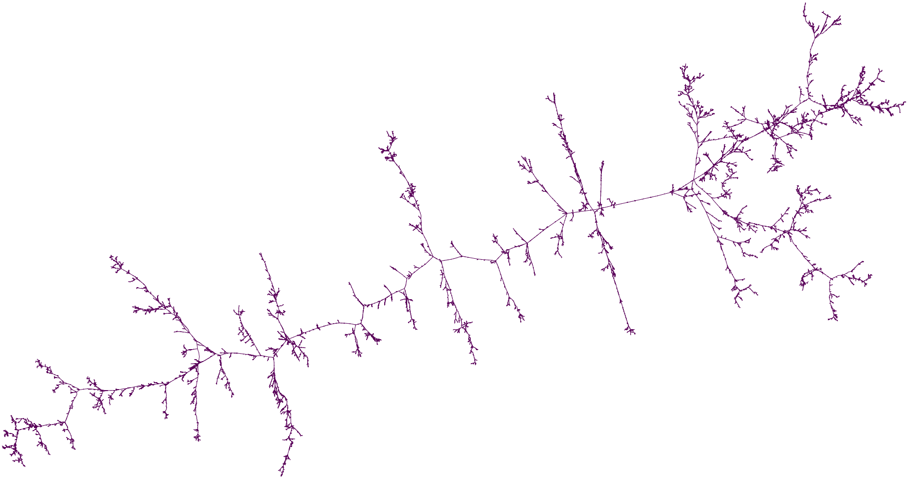


The Continuum Random Tree



CRT

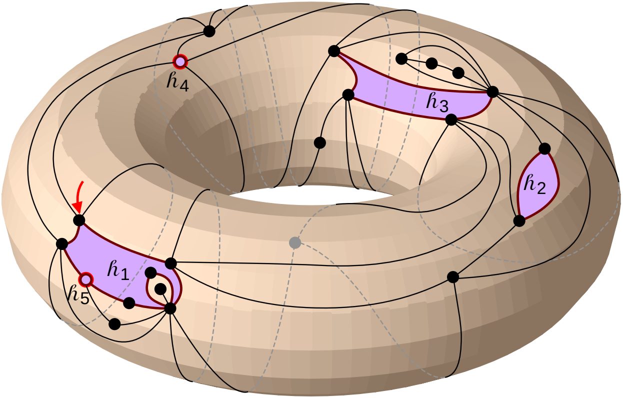
30 000 edges



Quadrangulation with holes

Definition

A quad. with holes is a rooted bipartite map with distinguished **vertices or faces** $h_1, \ldots, h_p$ and whose nondistinguished faces are of degree 4.



Hole perimeter:

- 0 if vertex
- degree if face

Brownian surfaces

- $g \geq 0$, $p \geq 0$ fixed integers, $(g, p) \neq (0, 0)$
- $\ell_n^i / \sqrt{2n} \to L^i \in [0, \infty)$ for $1 \leq i \leq p$
- $\mathbf{q}_n$ uniform among $\mathbf{Q}^{[g]}_{n,(\ell_n^1,\dots,\ell_n^p)}$

Brownian surfaces

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genus

half-perimeters of the p holes

number of quadrangles

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genus

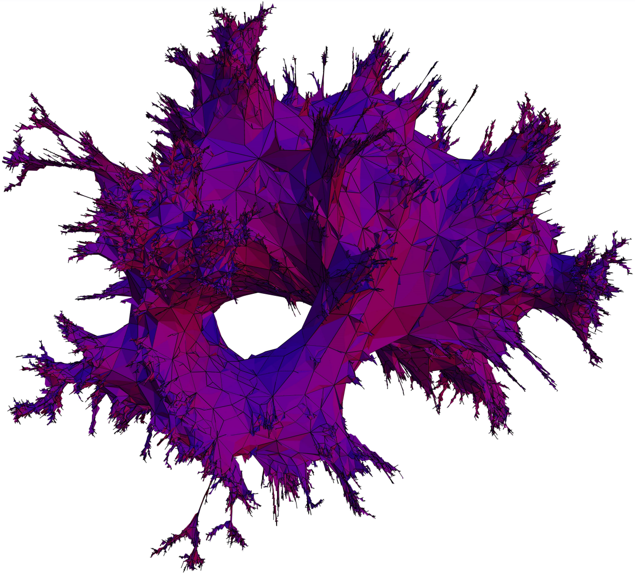
half-perimeters of the p holes

number of quadrangles

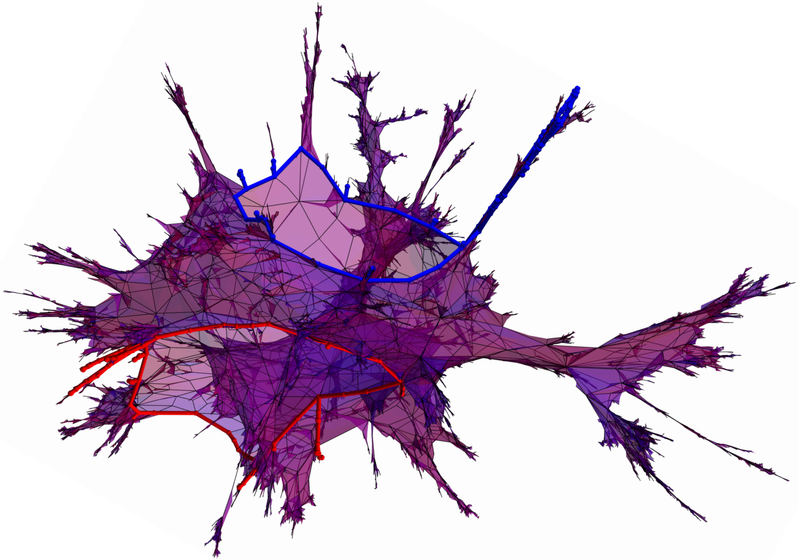
Theorem (B.–Miermont '22)

The sequence $\left((8n/9)^{-1/4}\mathbf{q}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space called the Brownian surface of genus g with boundary perimeter vector $(L^1, \ldots, L^p)$ (and unit area).

50 000 faces, genus 1



10 000 faces, genus 1, boundary lengths 60 and 80



Marks and measures

- In metric space $(\mathcal{Z}, d_{\mathcal{Z}})$, set $A^\varepsilon = \{z \in \mathcal{Z} : \inf_{a\in A} d_{\mathcal{Z}}(z,a) < \varepsilon\}$.
- Hausdorff metric: $d^{\mathrm{H}}_{\mathcal{Z}}(A,B) = \inf\{\varepsilon > 0 : A \subseteq B^\varepsilon \text{ and } B \subseteq A^\varepsilon\}$.
- Prokhorov metric: $d^{\mathrm{P}}_{\mathcal{Z}}(\mu,\nu) = \inf\{\varepsilon > 0 : \text{for all closed } A \subseteq \mathcal{Z},$
  $\mu(A) \le \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \le \mu(A^\varepsilon) + \varepsilon\}$.
- Consider
  - $(\mathcal{X}, d_{\mathcal{X}})$: nonempty compact metric space,
  - $\mathbf{A}$: k-tuple of nonempty compact subsets of $\mathcal{X}$ (called marks),
  - $\boldsymbol{\mu}$: ℓ-tuple of finite Borel measures on $\mathcal{X}$.
- (k-marked, ℓ-measured) Gromov–Hausdorff–Prokhorov metric:

$$\mathrm{d}^{(k,\ell)}_{\mathrm{GHP}}\big((\mathcal{X}, d_{\mathcal{X}}, \mathbf{A}, \boldsymbol{\mu}), (\mathcal{Y}, d_{\mathcal{Y}}, \mathbf{B}, \boldsymbol{\nu})\big) = \inf_{\substack{\phi:\mathcal{X}\to\mathcal{Z}\\ \psi:\mathcal{Y}\to\mathcal{Z}}} \Big\{ d^{\mathrm{H}}_{\mathcal{Z}}(\phi(\mathcal{X}), \psi(\mathcal{Y})) \vee \max_{1\le i\le k} d^{\mathrm{H}}_{\mathcal{Z}}(\phi(A_i), \psi(B_i)) \vee \max_{1\le j\le \ell} d^{\mathrm{P}}_{\mathcal{Z}}(\phi_*\mu_j, \psi_*\nu_j) \Big\}$$

with inf. over isometric embeddings in common metric space $\mathcal{Z}$.

Marked measured GHP convergence

- $\mathbf{q}$: quadrangulation with holes $\hbar_1, \dots, \hbar_p$.
  - Metric space: $(V(\mathbf{q}), d_{\mathbf{q}})$.
  - Marks: $\partial\mathbf{q} = \big(V(\hbar_1), \dots, V(\hbar_p)\big)$ where $V(\hbar_i) = \{\hbar_i\}$ if $\hbar_i$ is a vertex, or the set of vertices incident to $\hbar_i$ if it is a face.
  - Measures: $\mu_{\mathbf{q}} = \sum_{v \in V(\mathbf{q})} \delta_v$, and $\boldsymbol{\nu}_{\partial\mathbf{q}}$ where $\nu_{\partial\mathbf{q},i} = \sum_{v \in V(\hbar_i)} \delta_v$.
- $\mathbf{q}_n$ uniform among $\mathbf{Q}^{[g]}_{n,(\ell_n^1,\dots,\ell_n^p)}$, where $\ell_n^i/\sqrt{2n} \to L^i \in [0,\infty)$.

Theorem (B.–Miermont '22)

The sequence

$$\left(V(\mathbf{q}_n), \Big(\frac{9}{8n}\Big)^{1/4} d_{\mathbf{q}_n}, \partial\mathbf{q}_n, \frac{1}{n}\mu_{\mathbf{q}_n}, \frac{1}{\sqrt{8n}}\boldsymbol{\nu}_{\partial\mathbf{q}_n}\right)$$

converges weakly in the sense of the p-marked $p+1$-measured Gromov–Hausdorff–Prokhorov topology.

Topology and Hausdorff dimension

Theorem (B. '16)

Almost surely, the Brownian surface of genus g with boundary perimeter vector $(L^1, \dots, L^p)$ is homeomorphic to the surface of genus g with p' holes, where

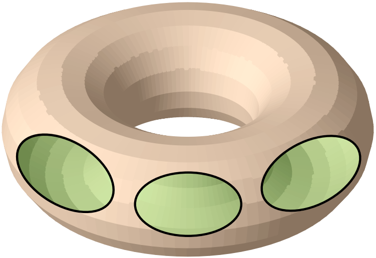
$$p' = \left|\{i : L^i > 0\}\right|.$$

Theorem (B. '16)

Almost surely, its Hausdorff dimension is 4 and that of each of its boundary components is 2.

Remark

The marks corresponding to indices i with $L^i = 0$ are singletons.



genus 1 with 3 holes

Toward Brownian nonorientable surfaces

- $\mathbf{q}_n$ uniform among quadrangulations with n faces of a fixed nonorientable surface

Theorem (Chapuy & Dołęga '17)

Up to extraction, the sequence $\left((8n/9)^{-1/4}\mathbf{q}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space.

A history of bijections

Encoding of pointed maps

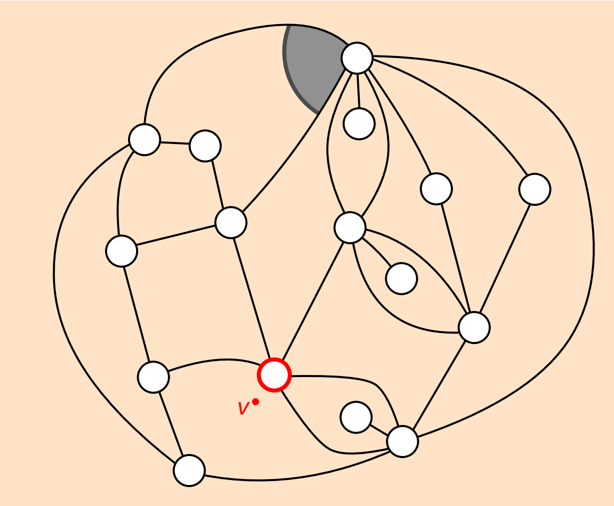
	sphere	orientable	nonorientable
bip. quad.	CVS	CMS	CD
general maps	BDG	BDG + CMS	B

CVS: [Cori–Vauquelin ’81] and [Schaeffer ’98]
BDG: [Bouttier–Di Francesco–Guitter ’04]
CMS: [Chapuy–Marcus–Schaeffer ’09]
CD: [Chapuy–Dołęga ’17]
B: [B. ’22]

Similar bijections

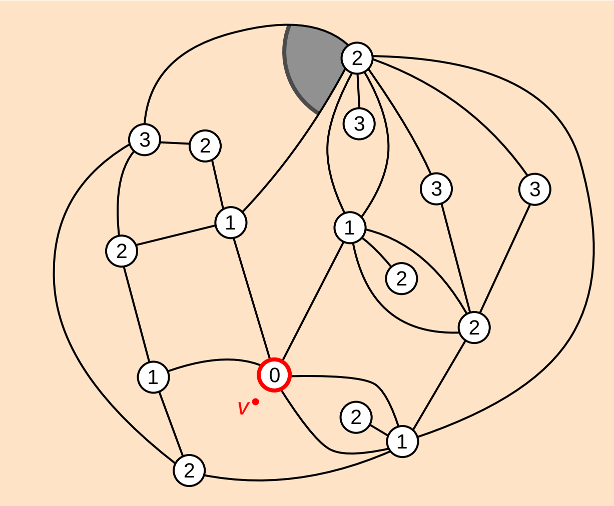
- [Miermont ’09] multi-pointed quadranguations
- [Ambjørn–Budd ’13] CVS with rules inverted

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



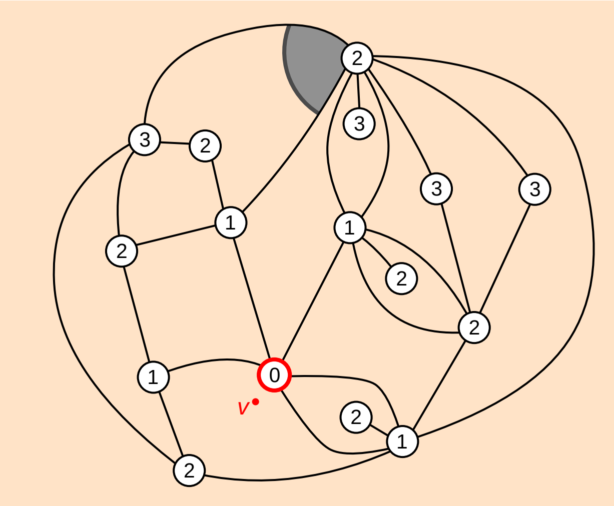
- Start with a pointed bipartite quadrangulation.

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer

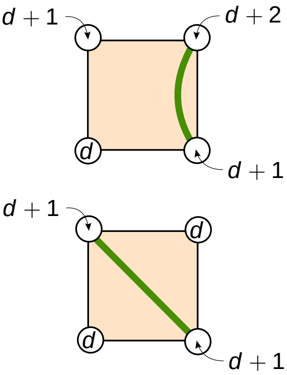


- Start with a pointed bipartite quadrangulation.
- Label the vertices with their distance to $v^\bullet$.

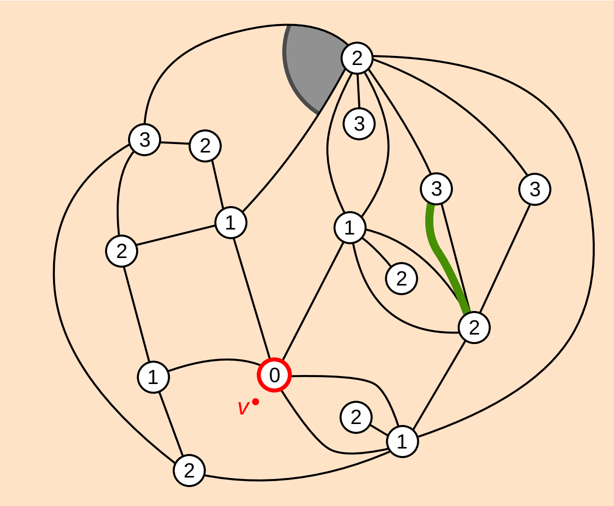
Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



- Apply the rule:



Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



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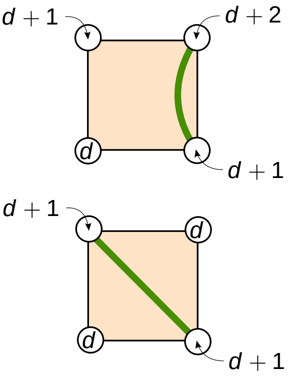
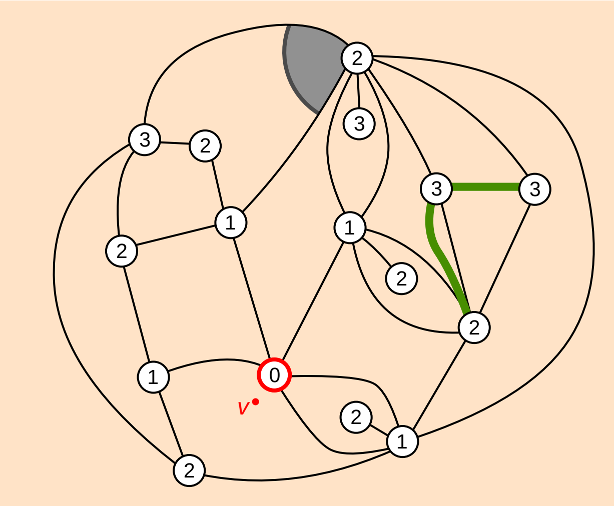


Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



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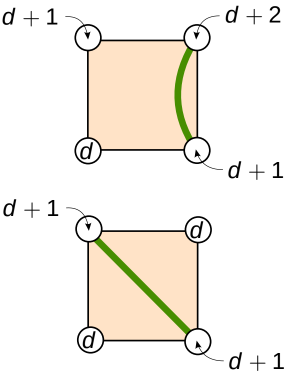
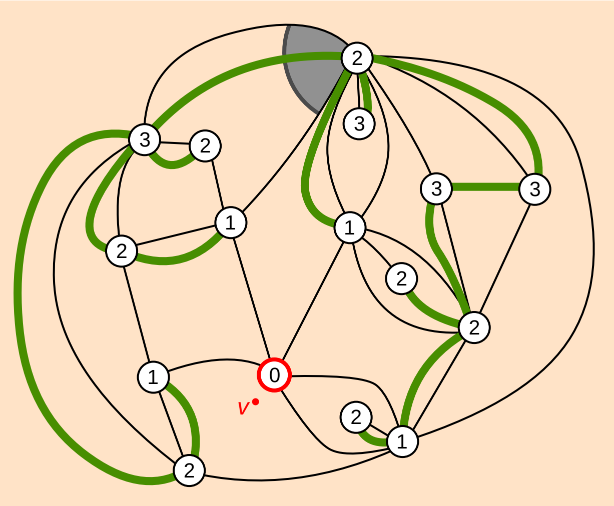


Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



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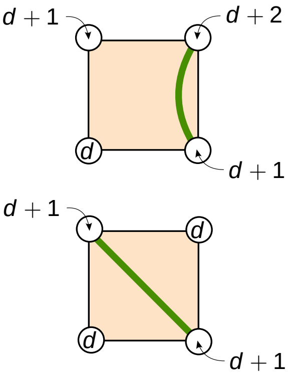
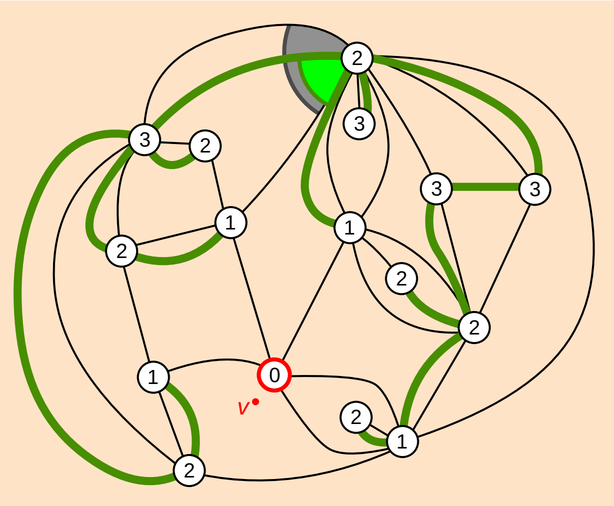
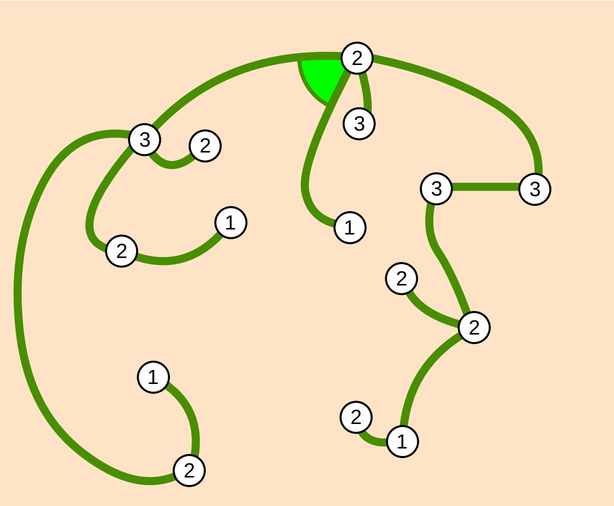


Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



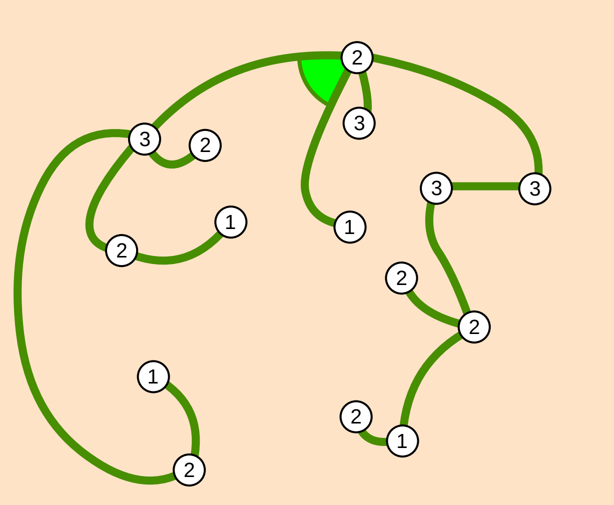
- Start with a pointed bipartite quadrangulation.
- Label the vertices with their distance to $v^{\bullet}$.
- Apply the rule.
- Root.

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



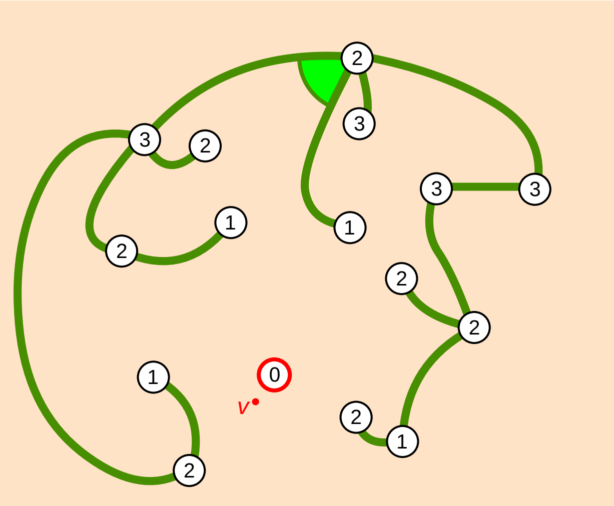
- Start with a pointed bipartite quadrangulation.
- Label the vertices with their distance to $v^\bullet$.
- Apply the rule.
- Root.
- Remove the initial edges and $v^\bullet$.

Inverse construction



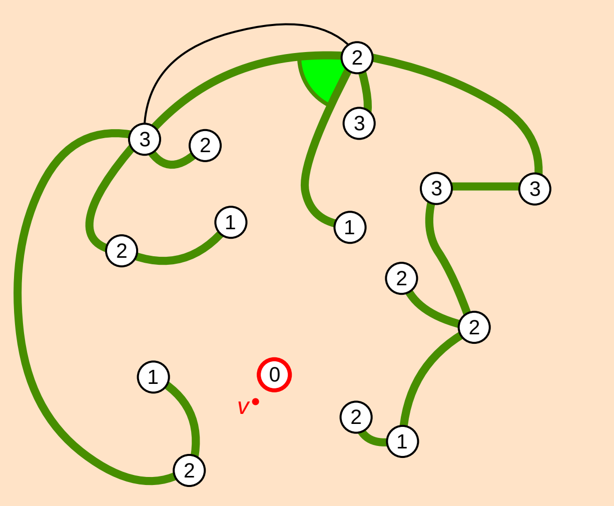
- Take a well-labeled unicellular map.

Inverse construction



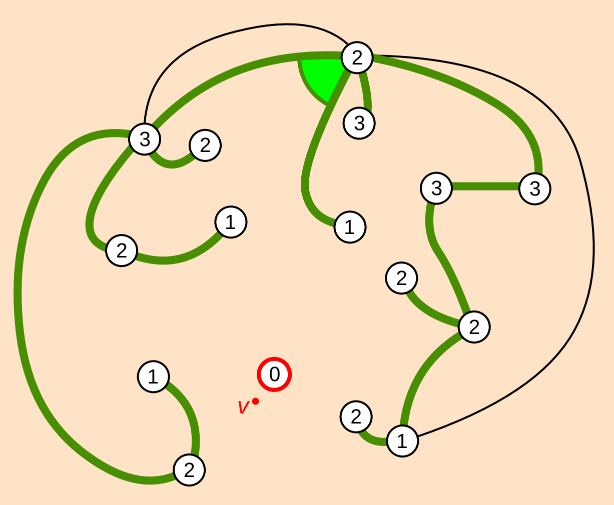
- Take a well-labeled unicellular map.
- Add a vertex $v^\bullet$ inside the unique face.

Inverse construction



- Take a well-labeled unicellular map.
- Add a vertex $v^\bullet$ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.

Inverse construction



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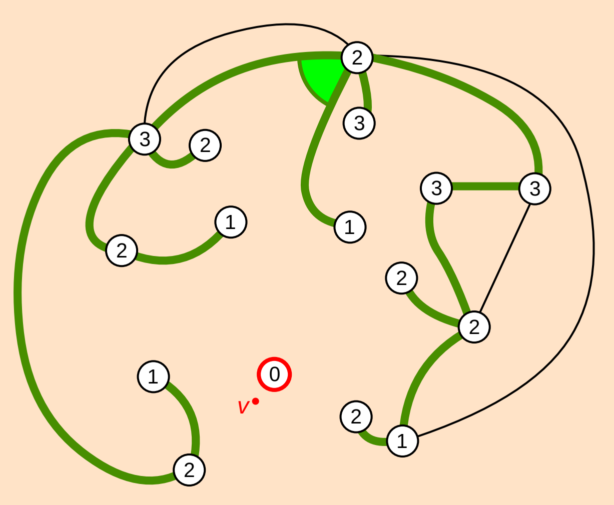

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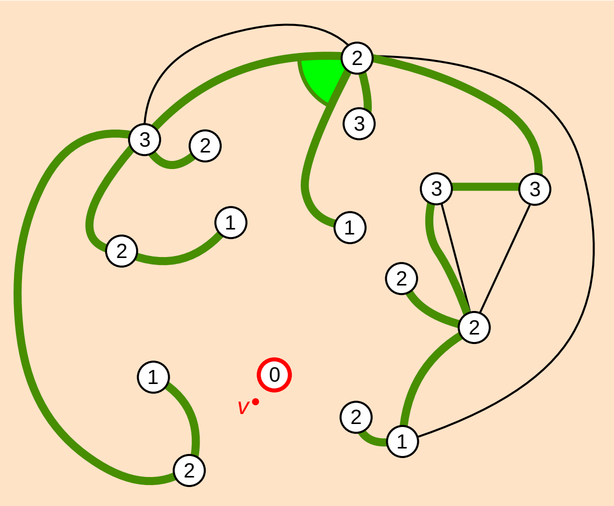

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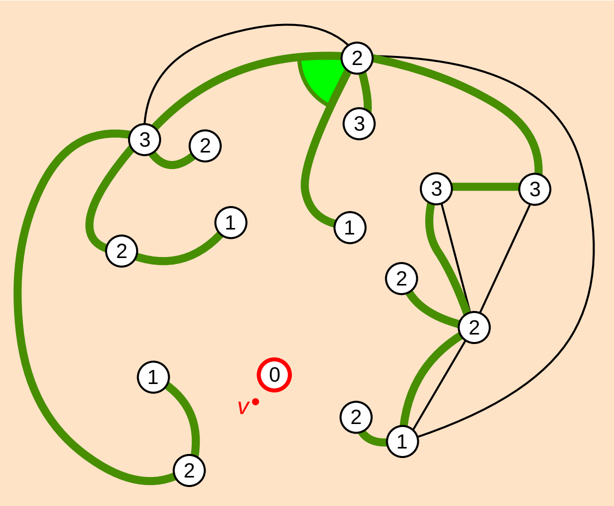

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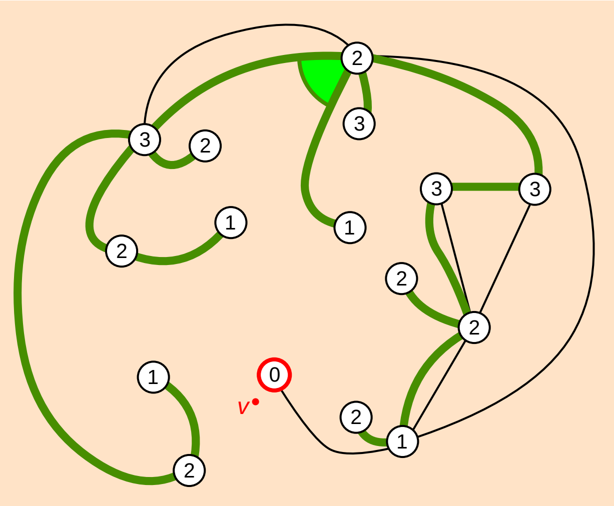

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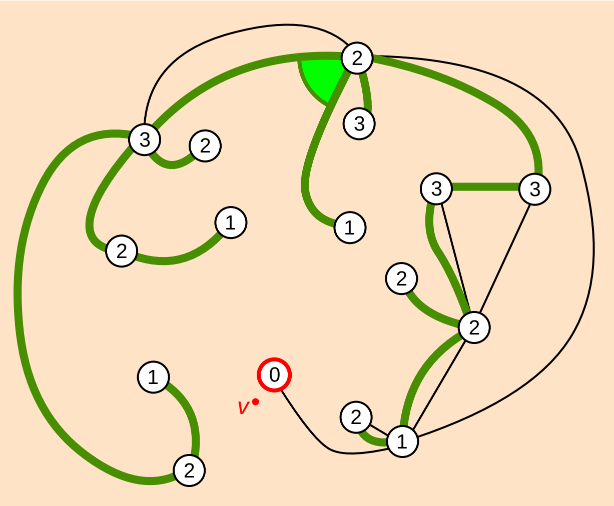

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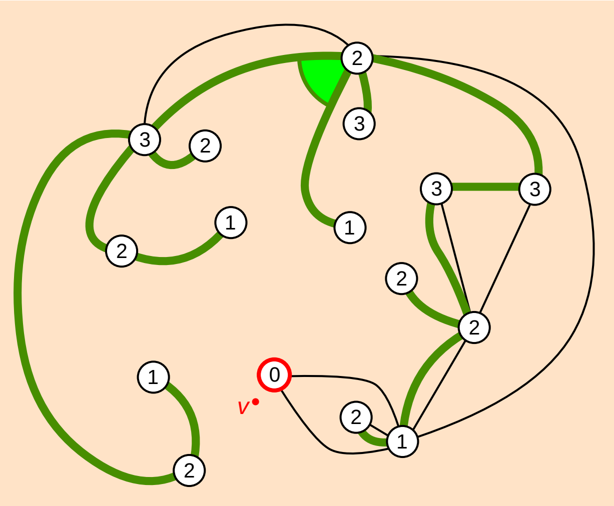

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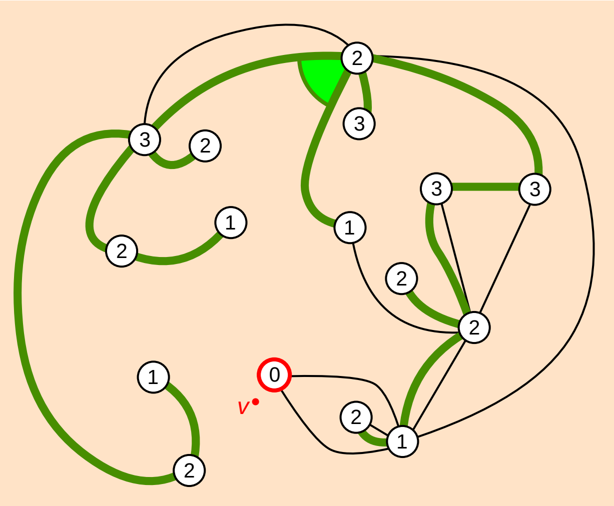

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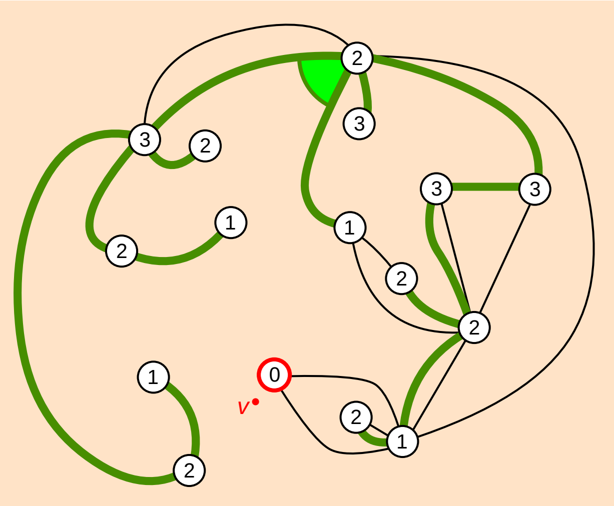

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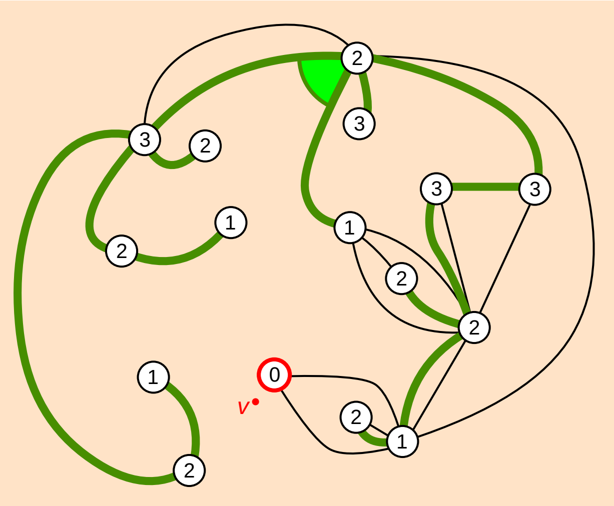

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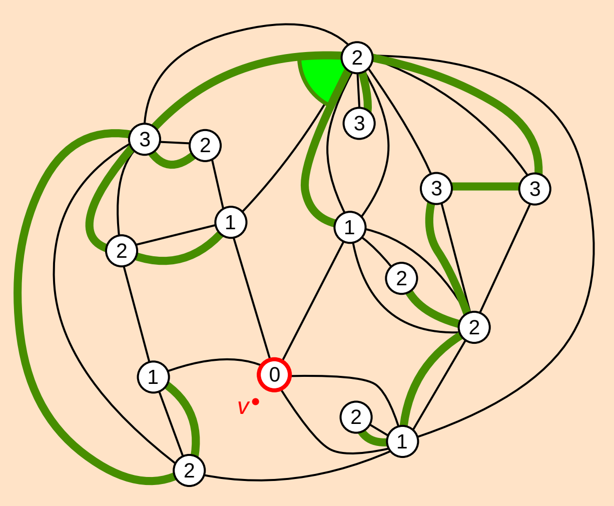
Inverse construction



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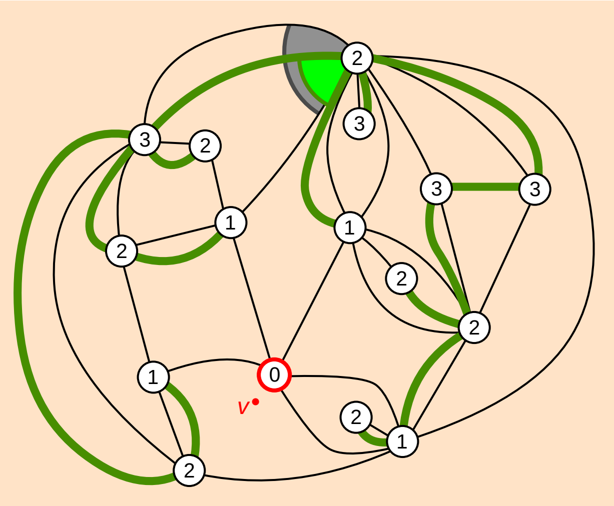
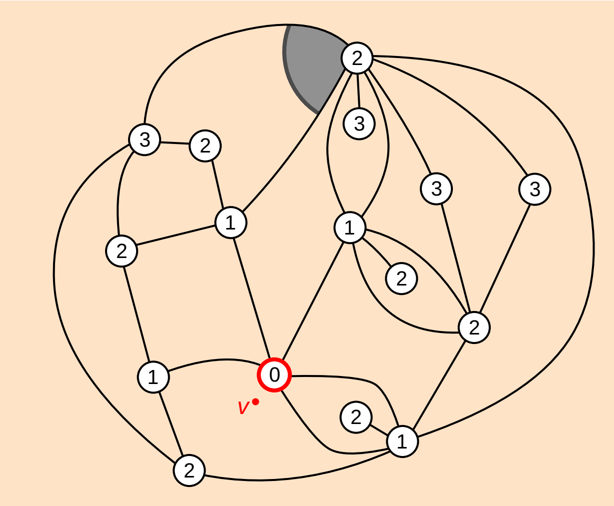
Inverse construction



- Take a well-labeled unicellular map.
- Add a vertex $v^\bullet$ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.
- Root and remove the initial edges.

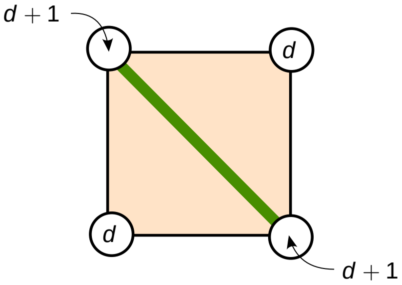
Inverse construction



- Take a well-labeled unicellular map.
- Add a vertex $v^\bullet$ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.
- Root and remove the initial edges.

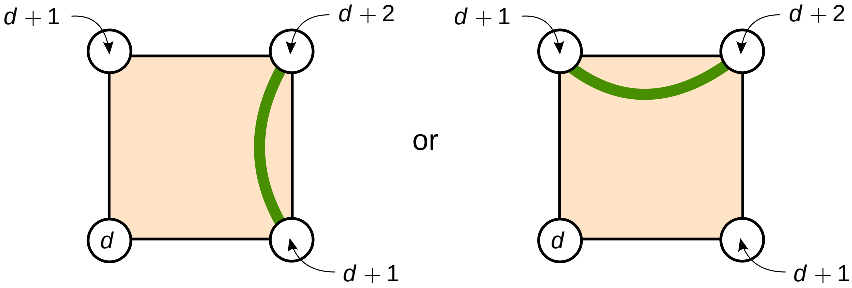
What could go wrong with nonorientable maps?

From quadrangulations to unicellular maps



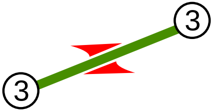
What could go wrong with nonorientable maps?

From quadrangulations to unicellular maps



What could go wrong with nonorientable maps?

From unicellular maps to quadrangulations



What could go wrong with nonorientable maps?

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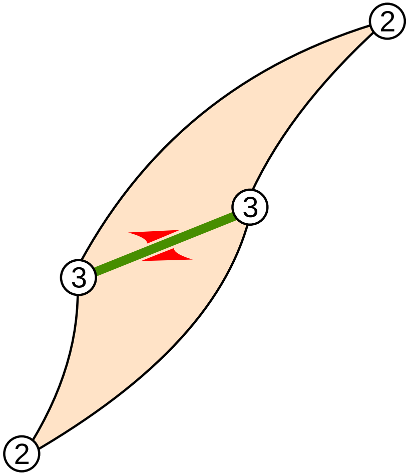
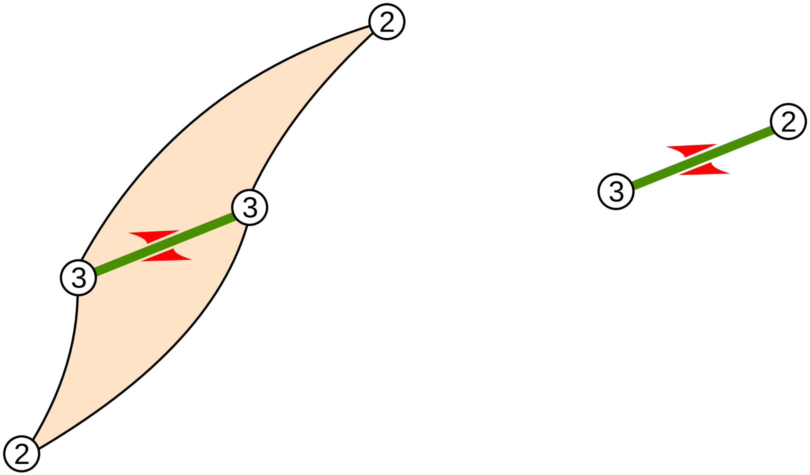


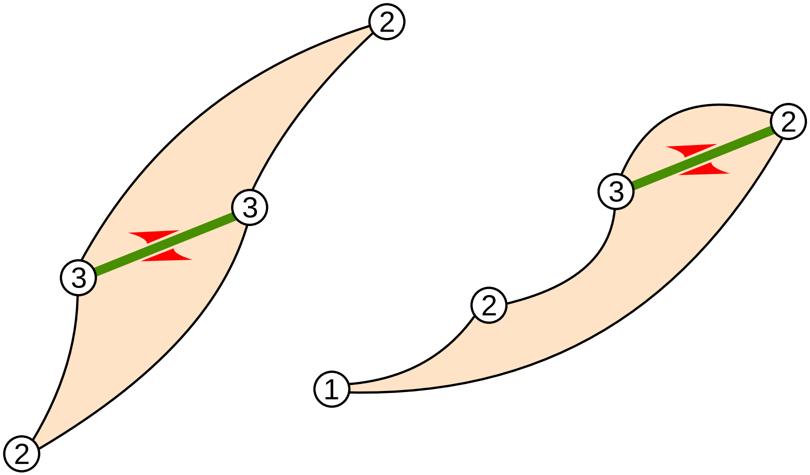
What could go wrong with nonorientable maps?

From unicellular maps to quadrangulations



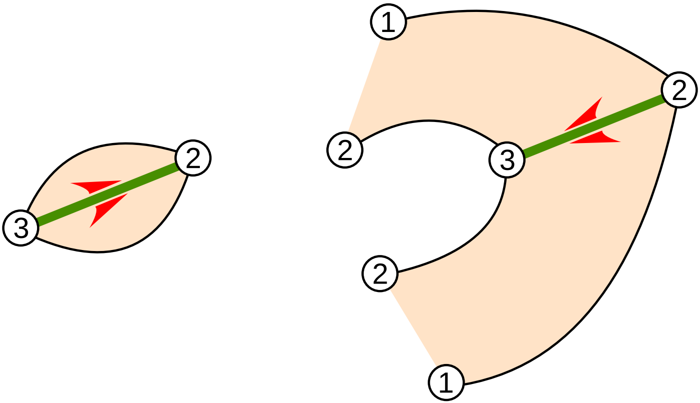
What could go wrong with nonorientable maps?

From unicellular maps to quadrangulations

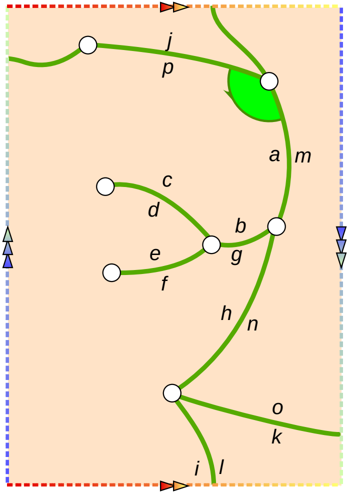


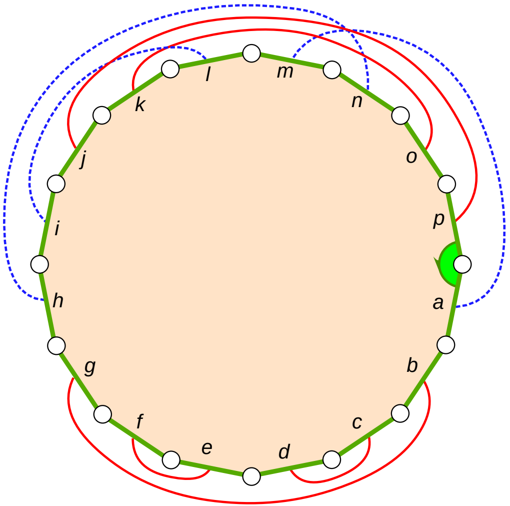
What could go wrong with nonorientable maps?

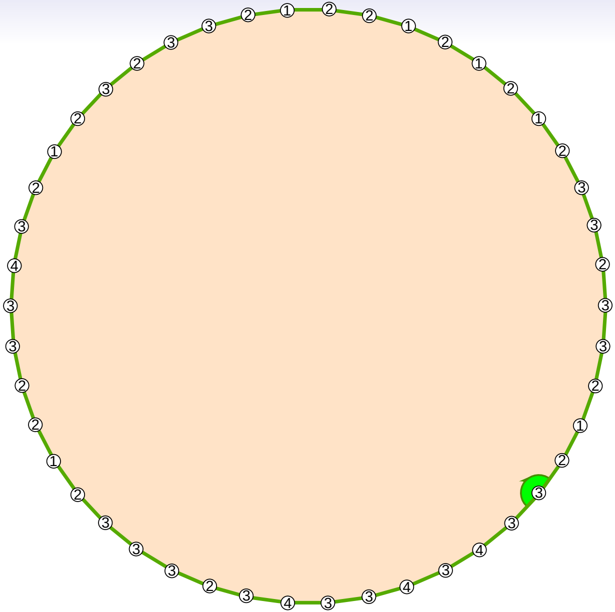
From unicellular maps to quadrangulations

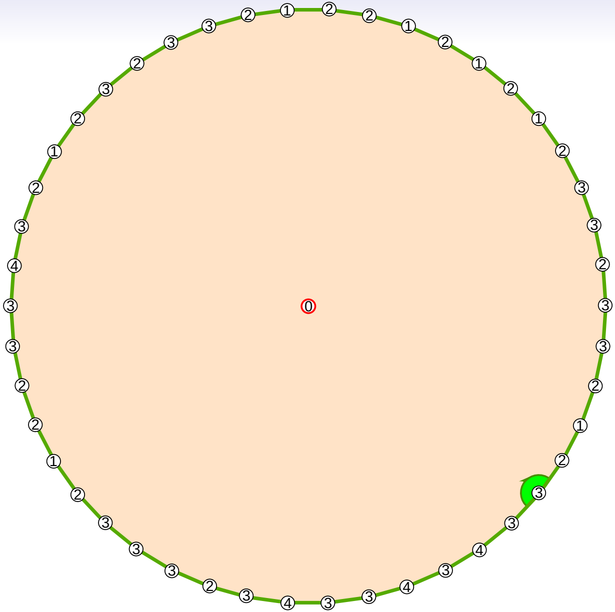


Unicellular maps seen as polygons with paired sides





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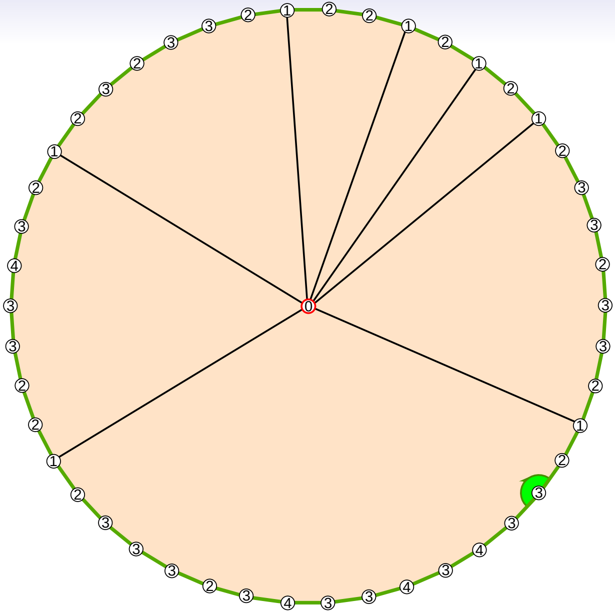
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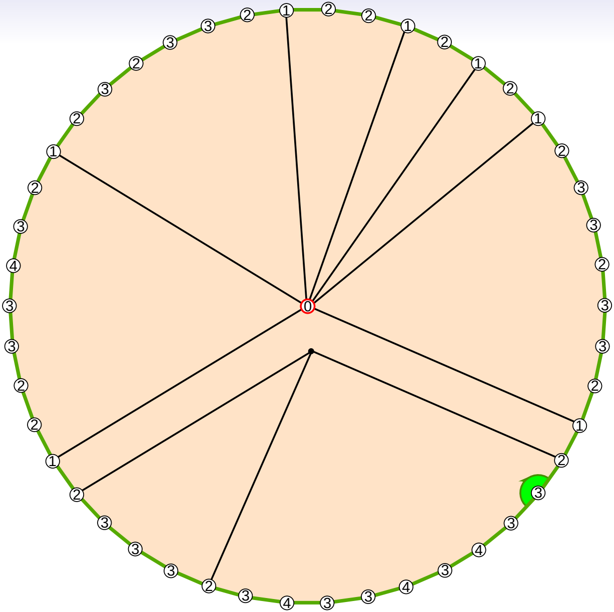

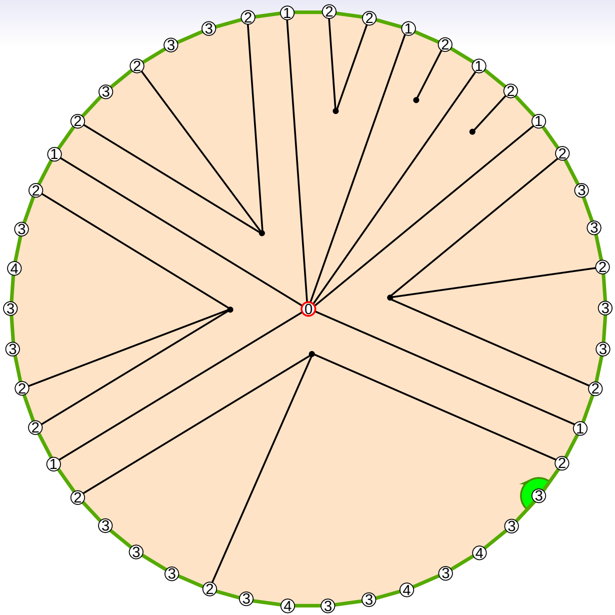
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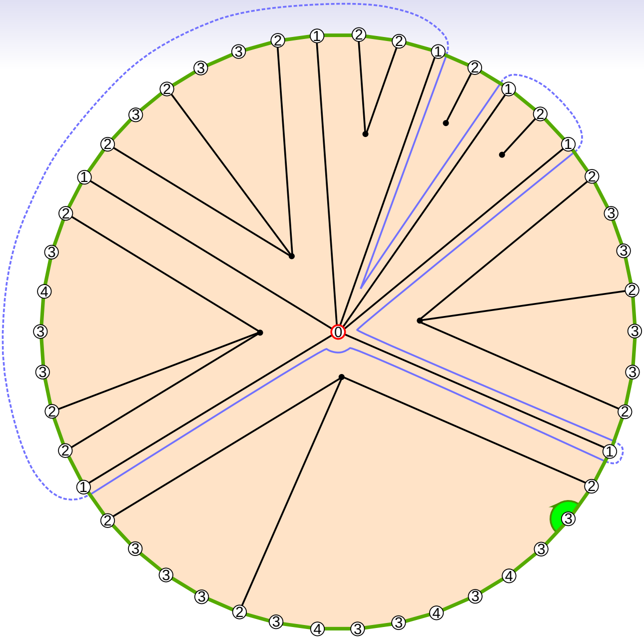
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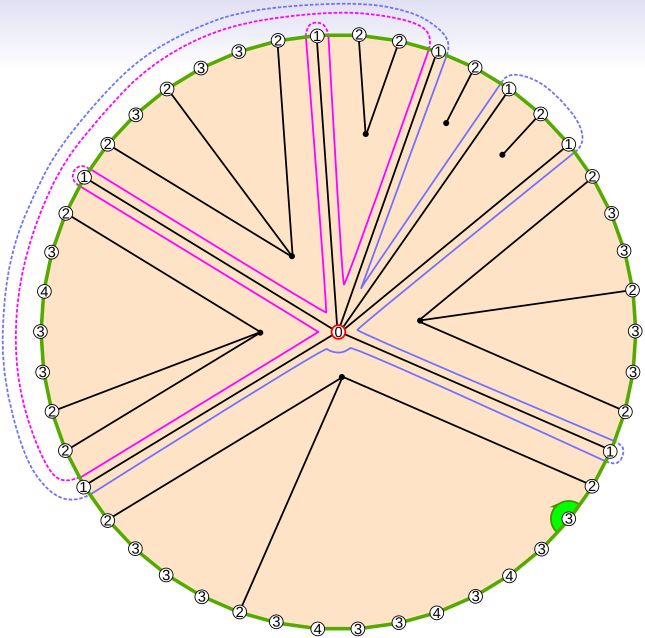
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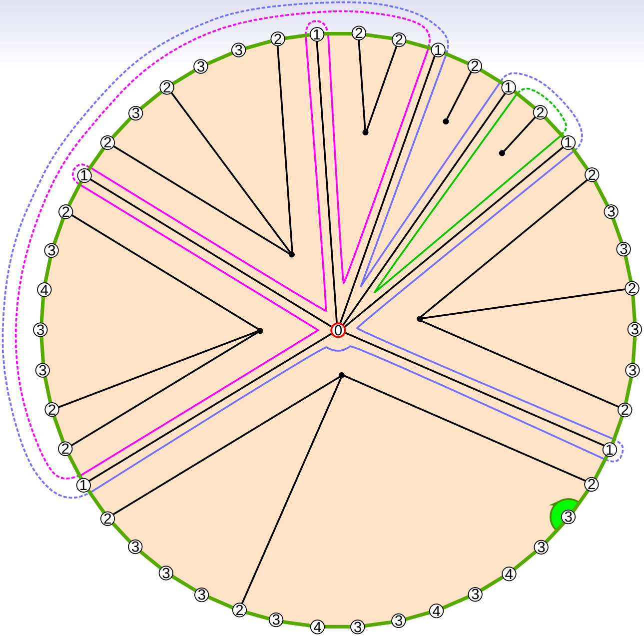
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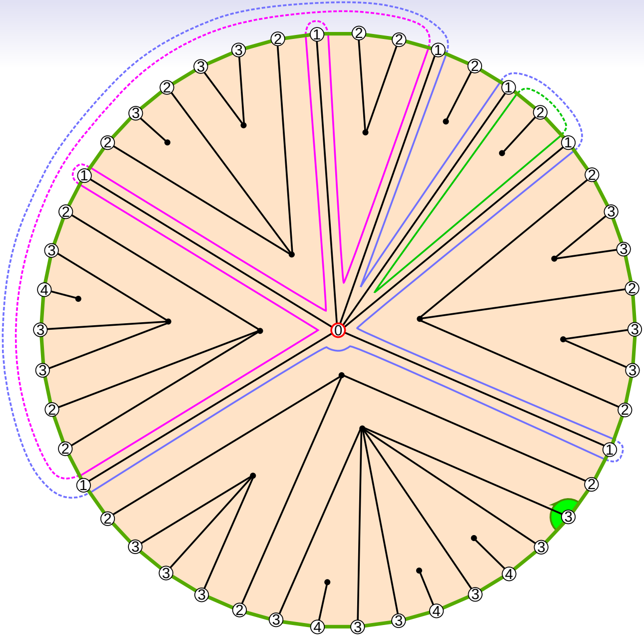

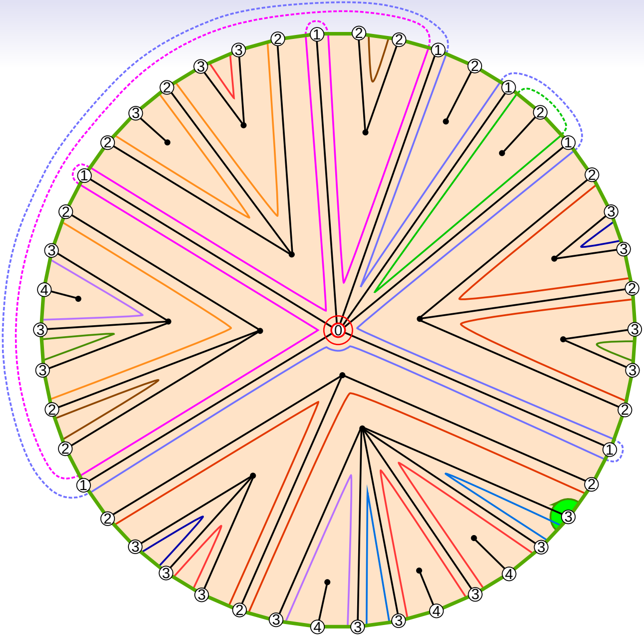

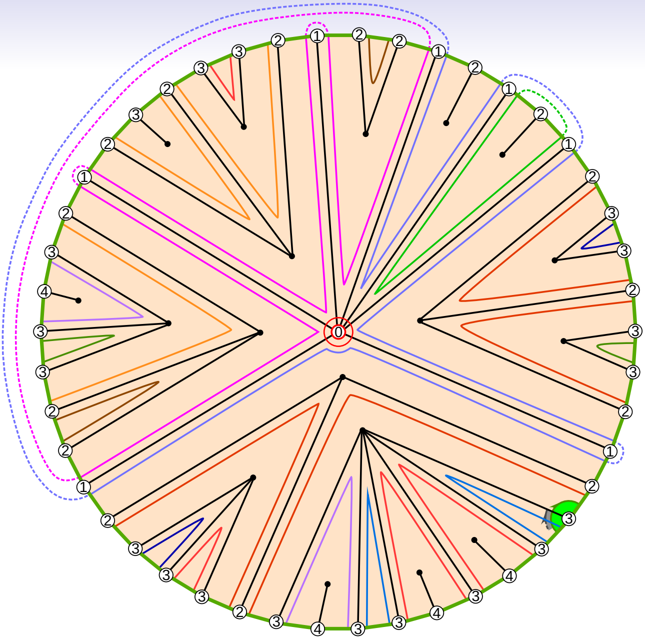

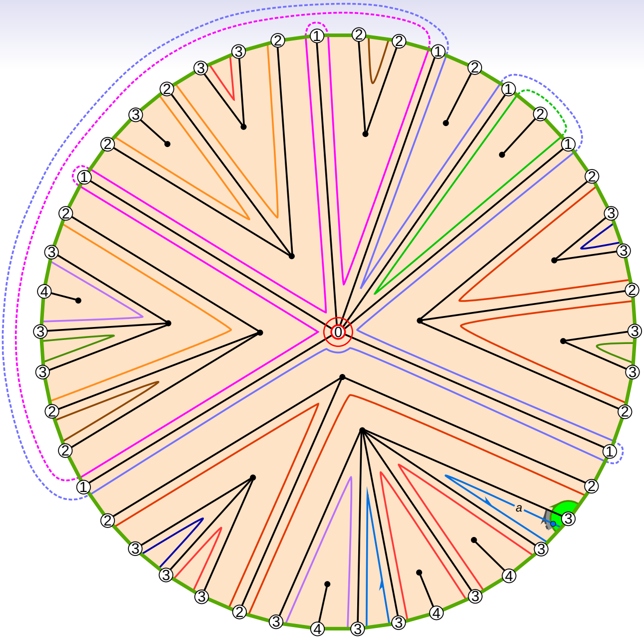

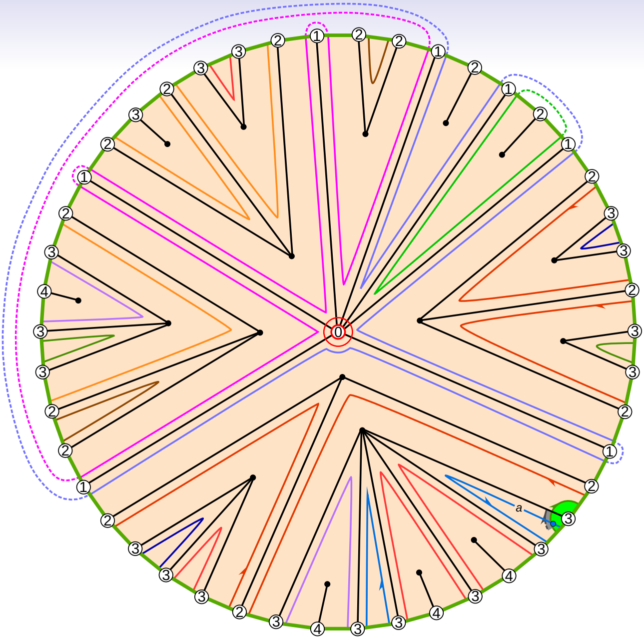

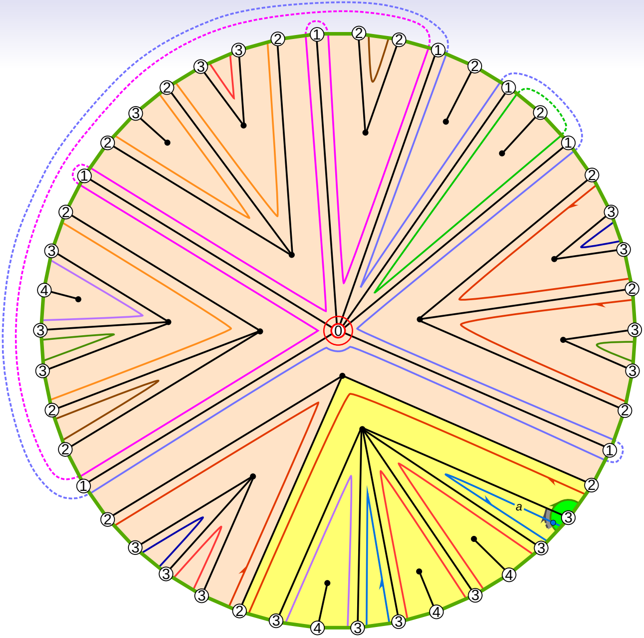

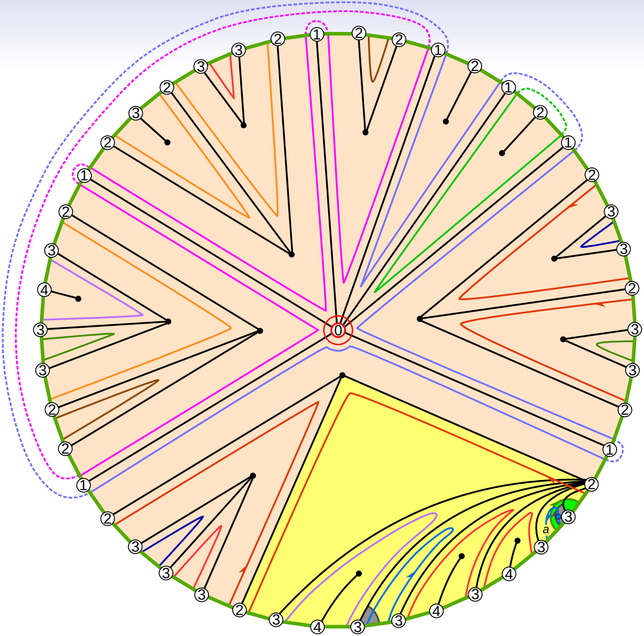
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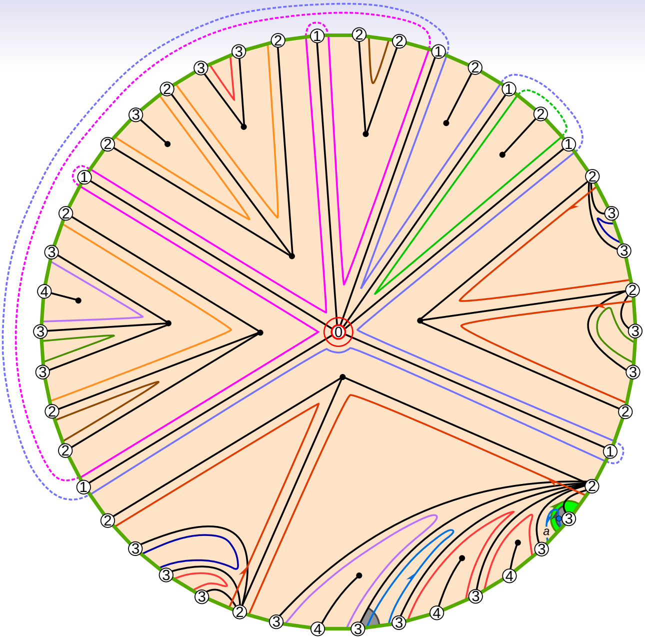

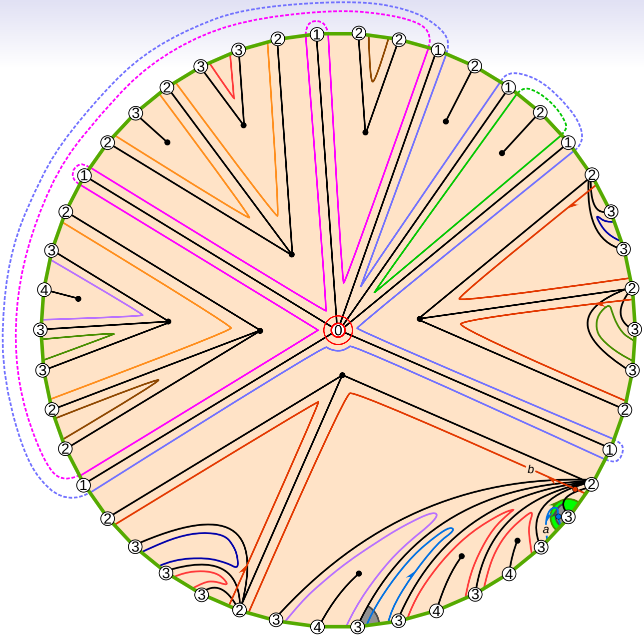
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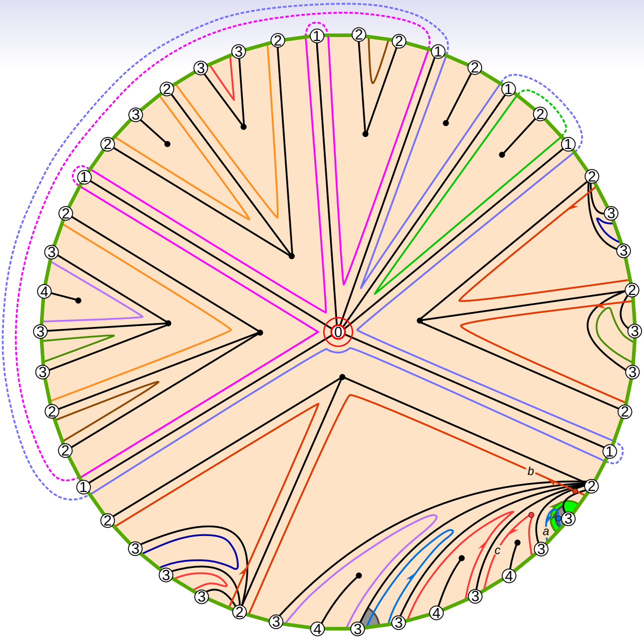

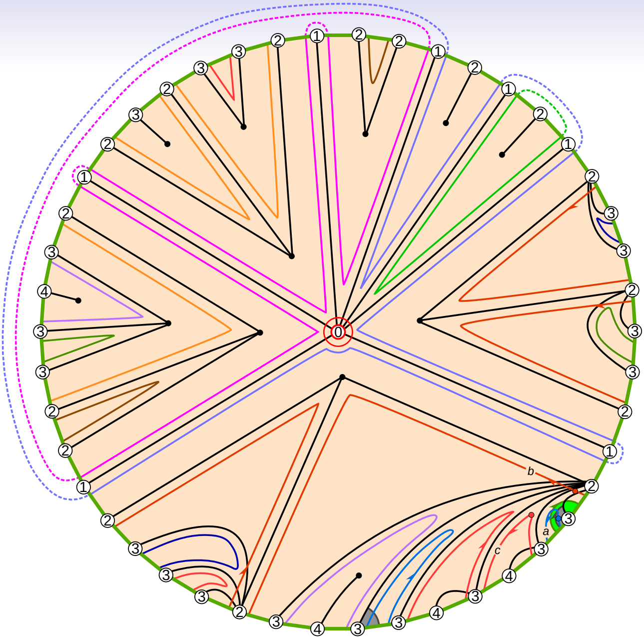
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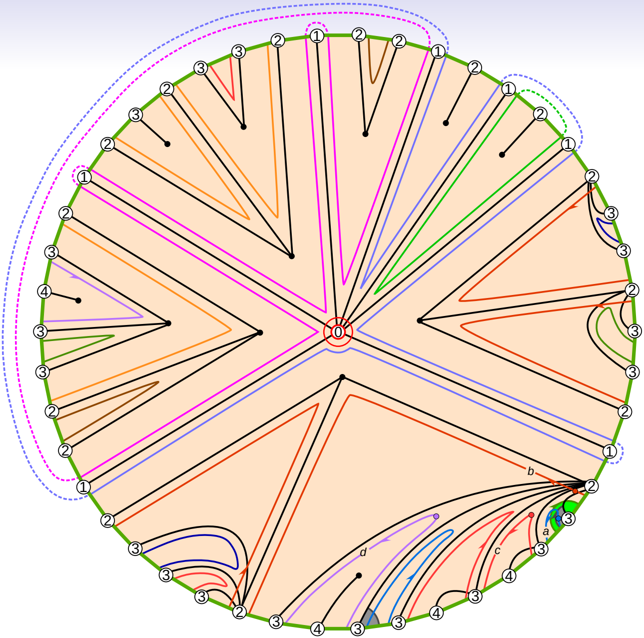
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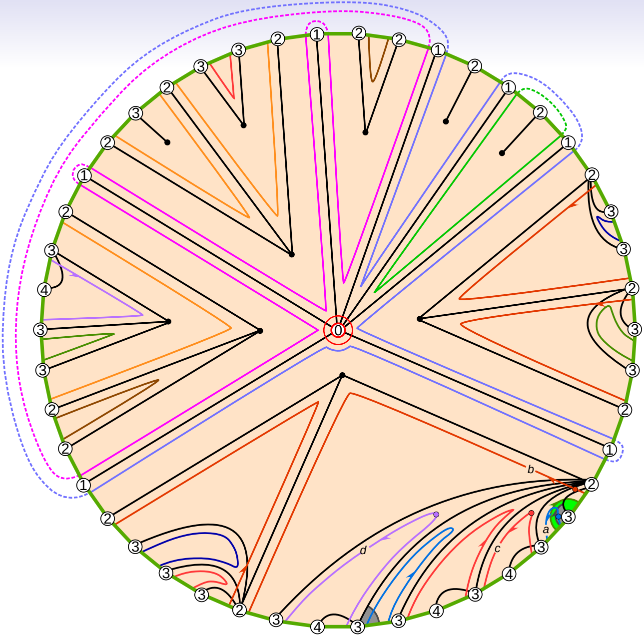
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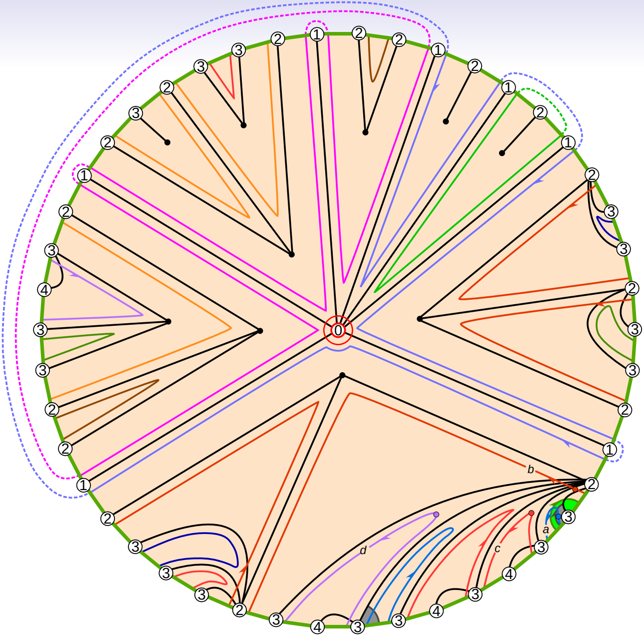
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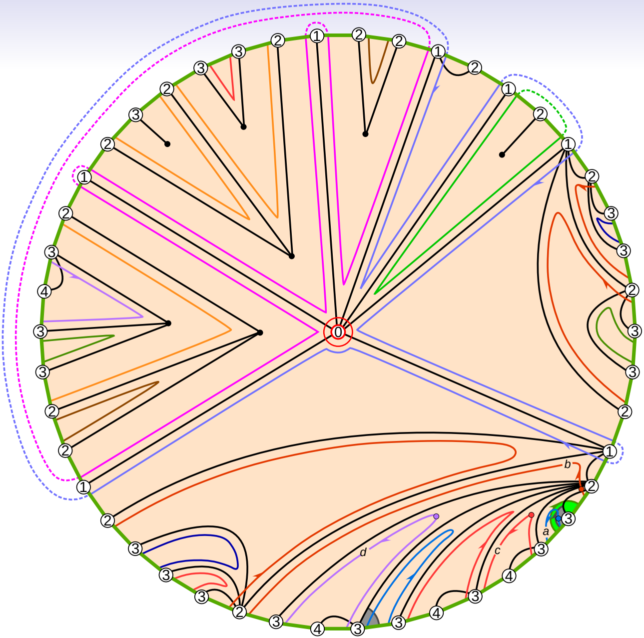
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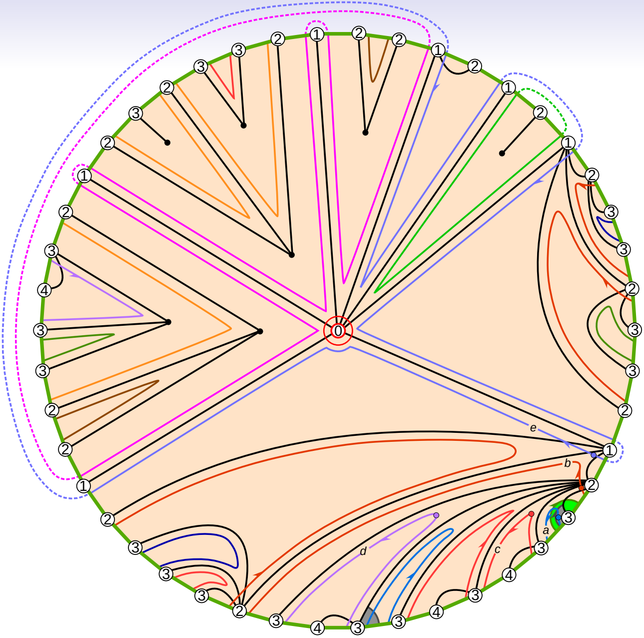

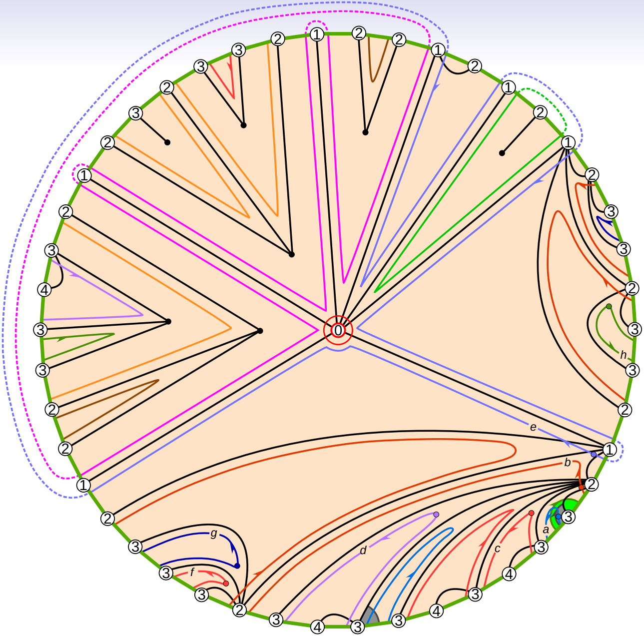
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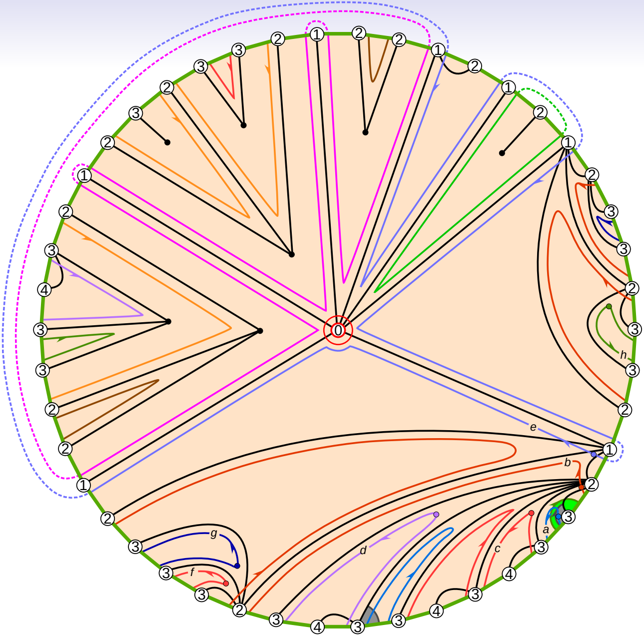
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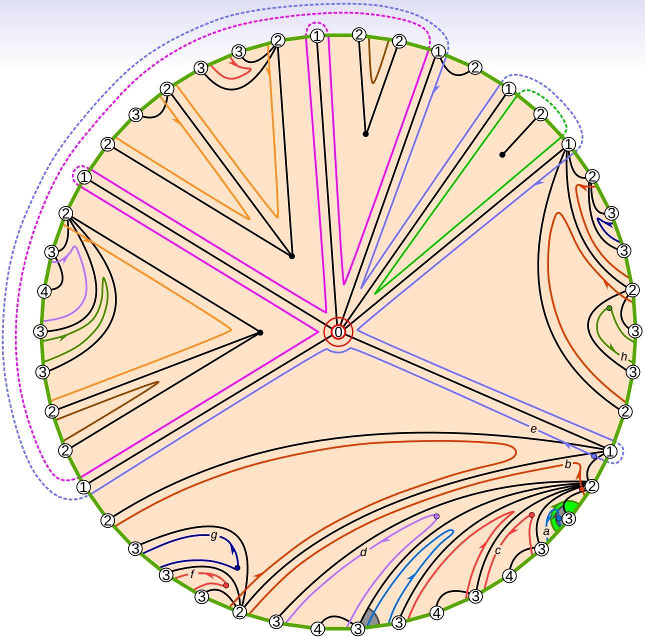
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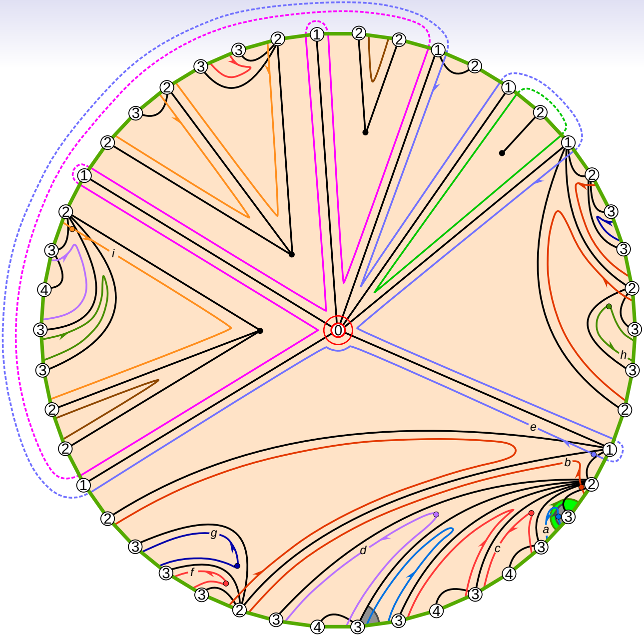
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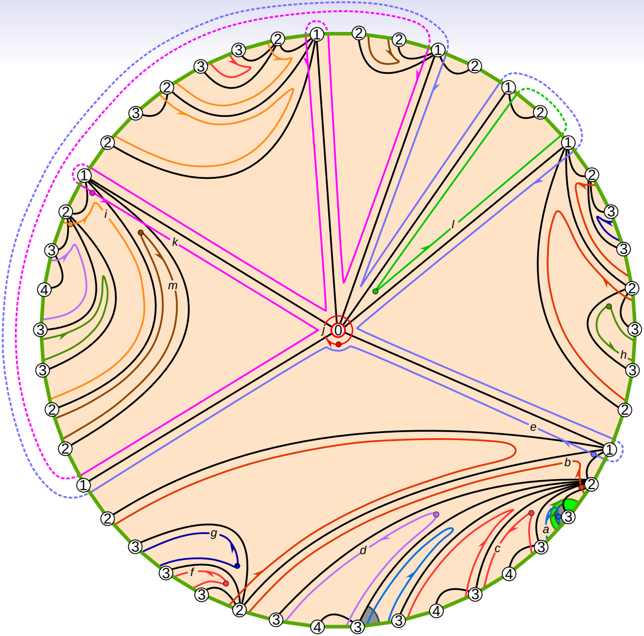
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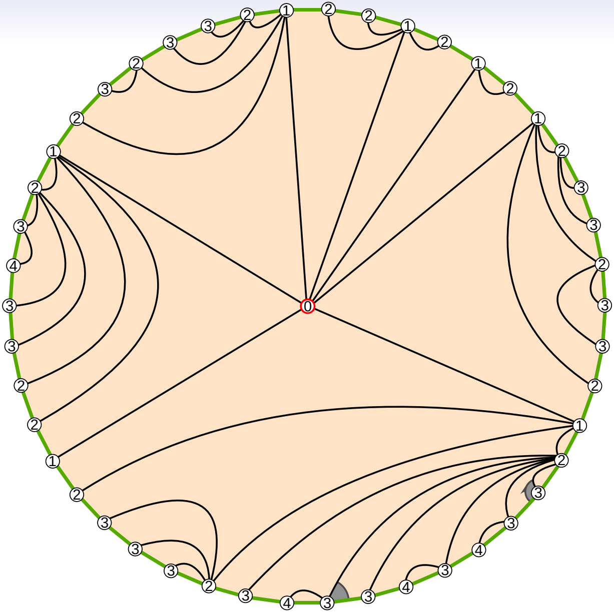
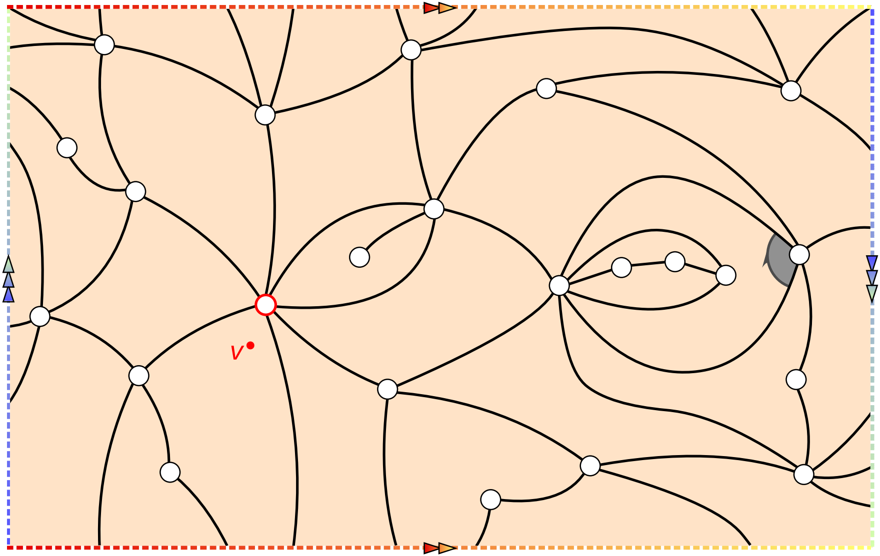


Chapuy–Dołęga (revisited)



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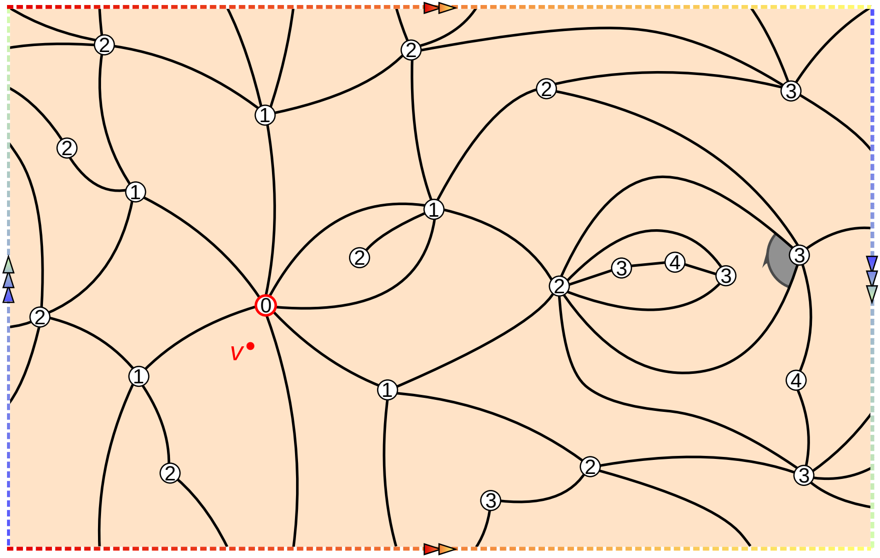

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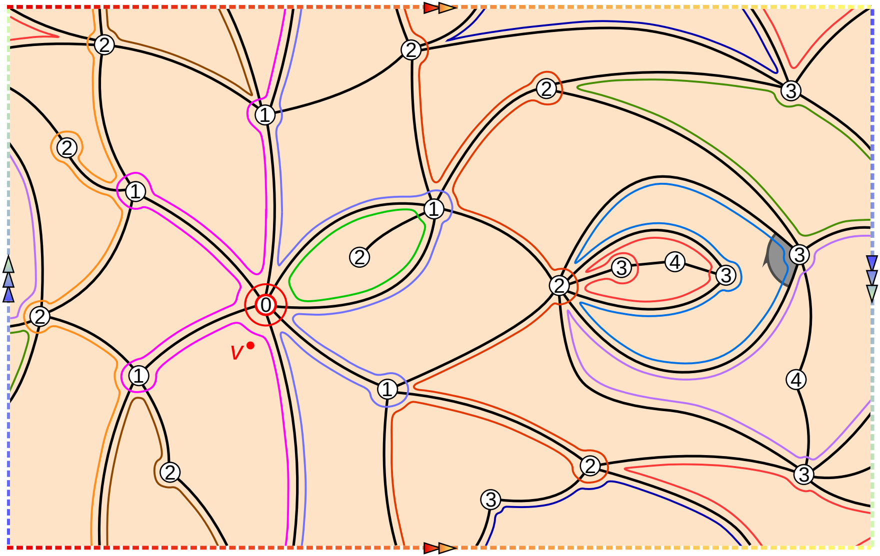

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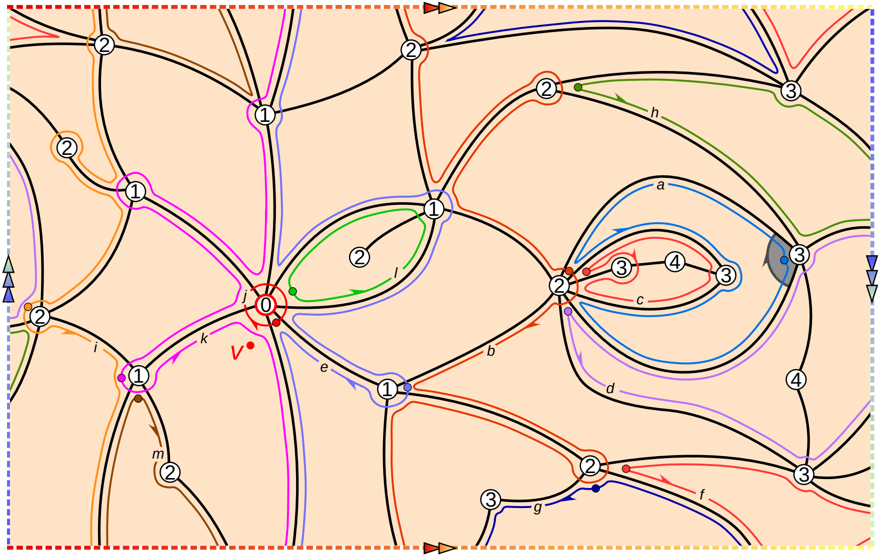

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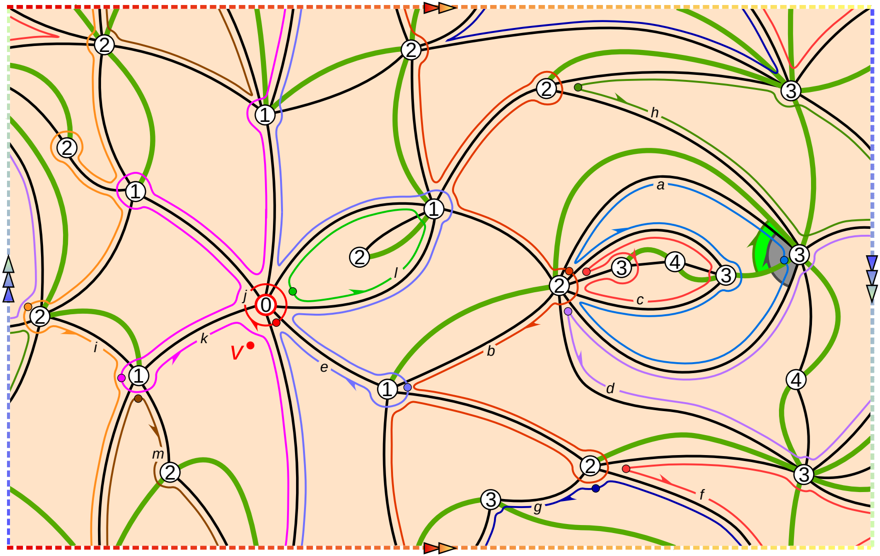

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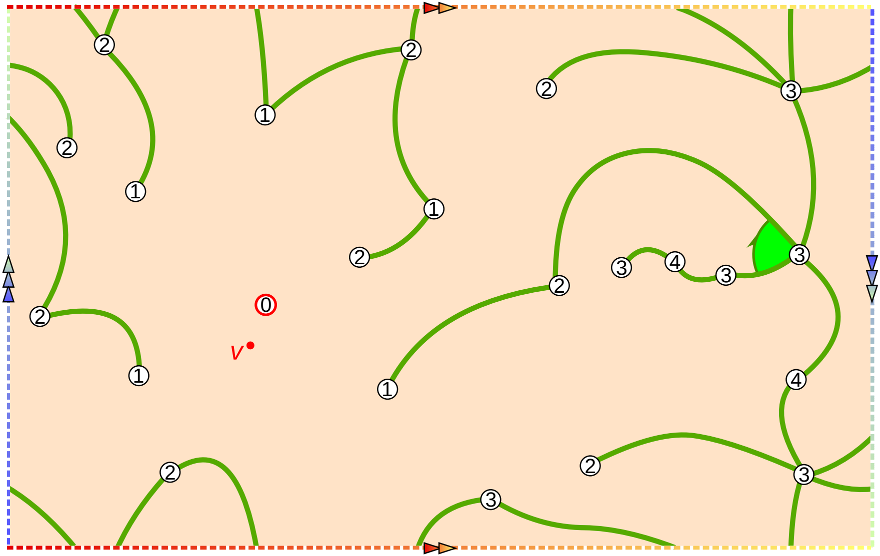
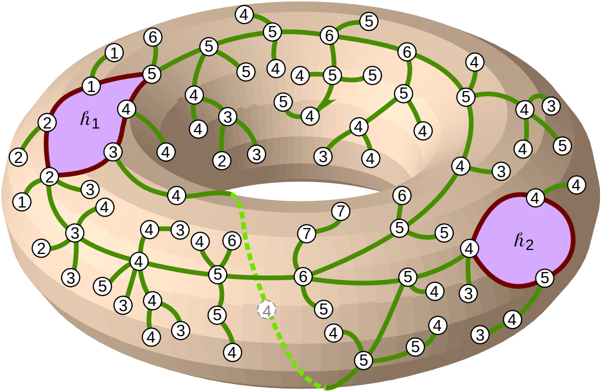
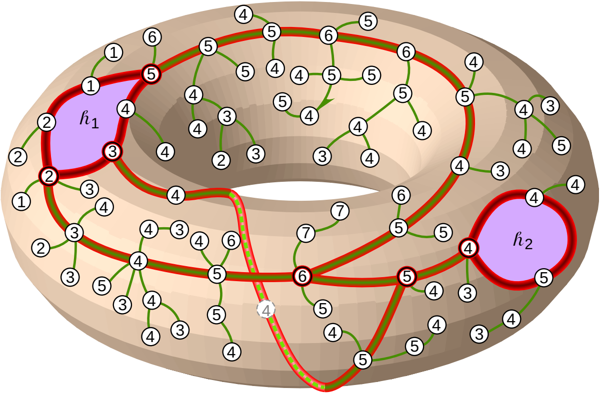
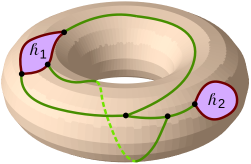


Scheme



Scheme





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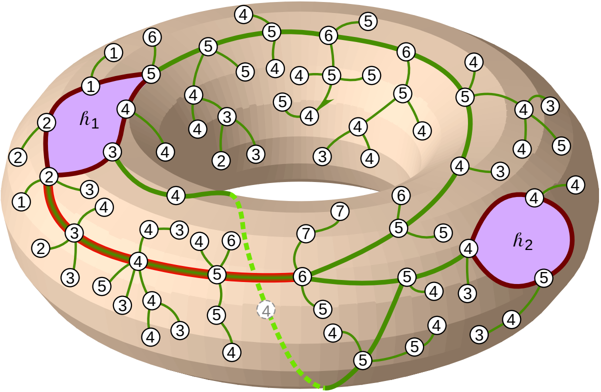
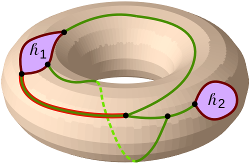
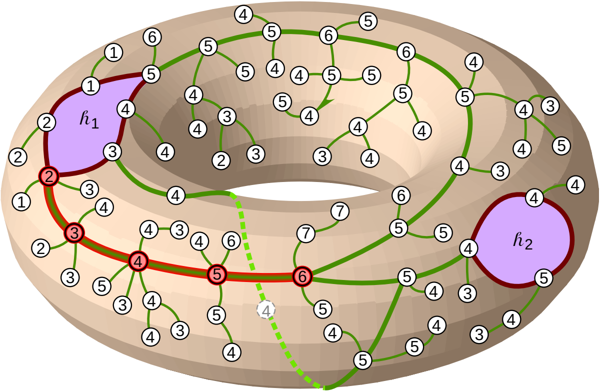


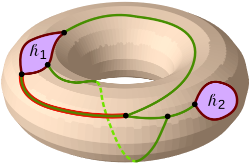


scheme

With each edge of the scheme, we associate:

Scheme



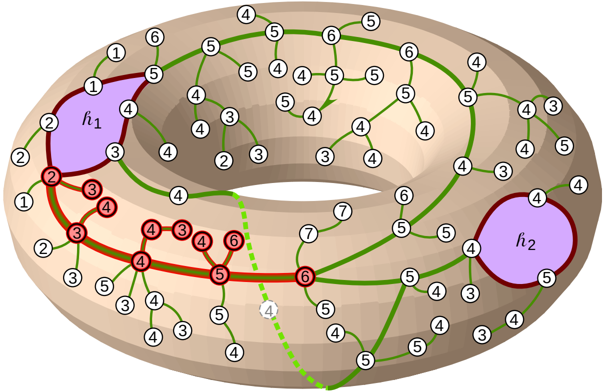


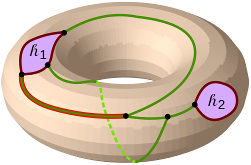
scheme

With each edge of the scheme, we associate:

- a Motzkin bridge

Scheme





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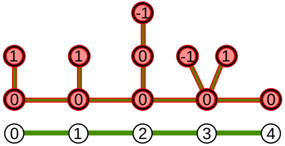
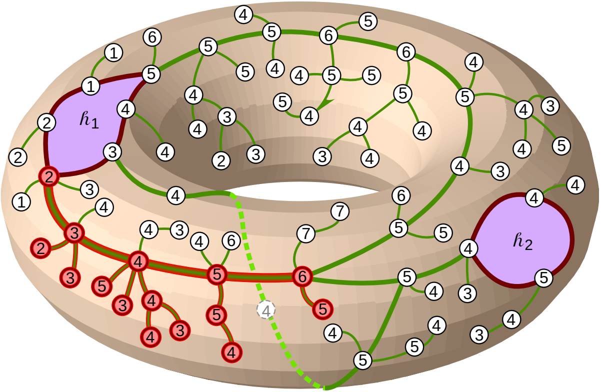
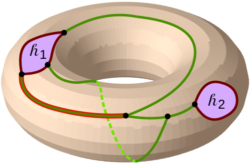


With each edge of the scheme, we associate:

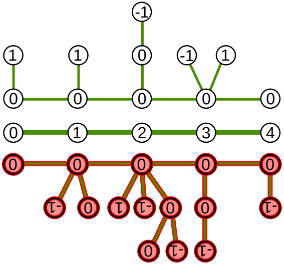
- a Motzkin bridge
- one or two well-labeled forests

Scheme





scheme

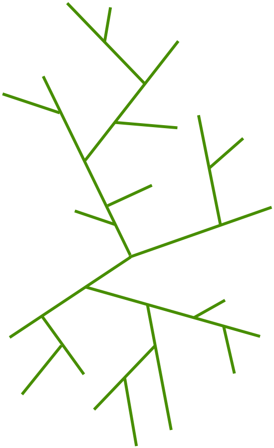


With each edge of the scheme, we associate:

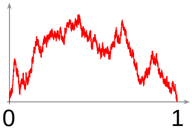
- a Motzkin bridge
- one or two well-labeled forests

Construction of the Brownian sphere $((g, p) = (0, 0))$

Recall how the Brownian sphere is constructed.

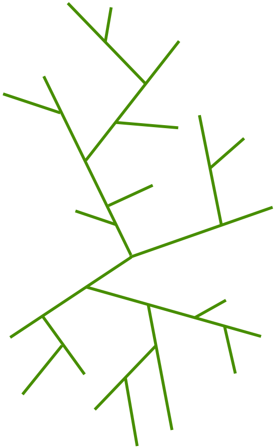


- Consider the CRT, that is, the random real tree encoded by the normalized Brownian excursion.



Construction of the Brownian sphere $((g, p) = (0, 0))$

Recall how the Brownian sphere is constructed.



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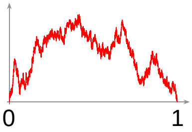
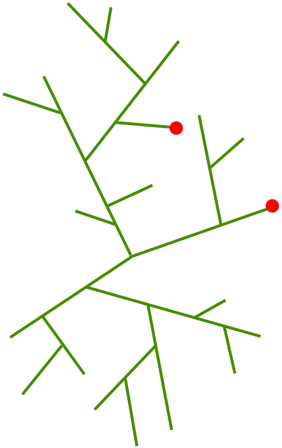


- Put Brownian labels Z on it.

Construction of the Brownian sphere $((g,p)=(0,0))$

Recall how the Brownian sphere is constructed.



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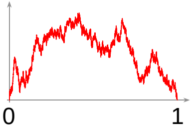
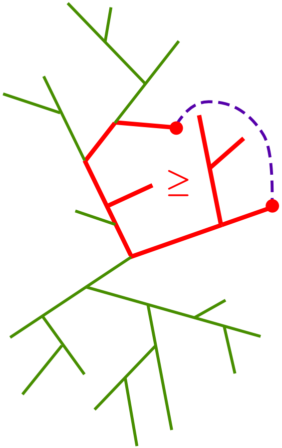


- Put Brownian labels Z on it.
- Identify the points a and b whenever $Z_a = Z_b = \min_{[a,b]} Z$.

Construction of the Brownian sphere ($(g,p)=(0,0)$)

Recall how the Brownian sphere is constructed.



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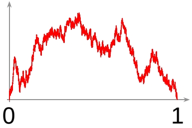

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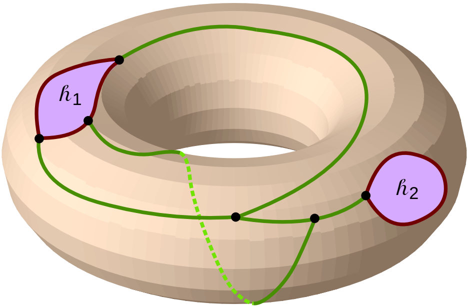

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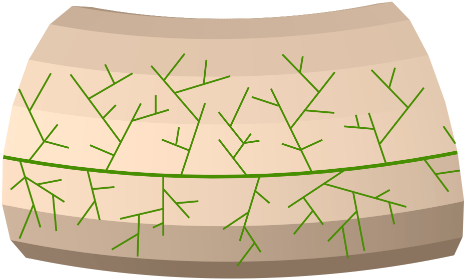

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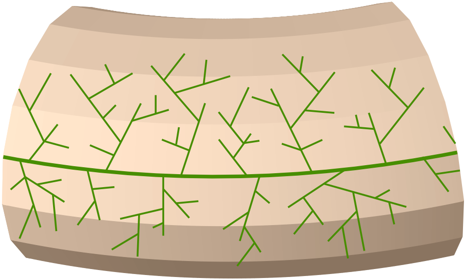

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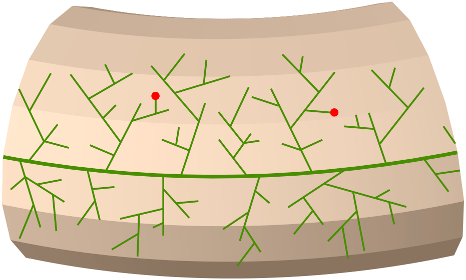

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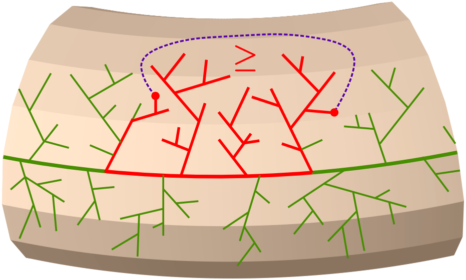

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Geodesic space

Definition

In a compact metric space $(\mathcal{X}, \delta)$, a geodesic from $x \in \mathcal{X}$ to $y \in \mathcal{X}$ is a continuous path $\wp : [0, \delta(x, y)] \to \mathcal{X}$ such that $\wp(0) = x$, $\wp(\delta(x, y)) = y$ and

$$\delta(\wp(s), \wp(t)) = |t - s| \qquad \text{for all} \qquad s, t \in [0, \delta(x, y)].$$

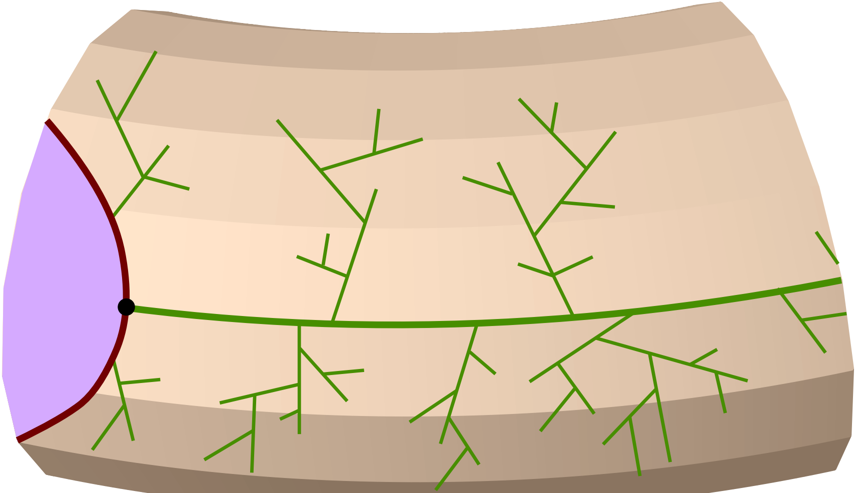
Definition

A geodesic space is a compact metric space in which every pair of points is connected by (at least) one geodesic.

Proposition

Every Brownian surface is a.s. a geodesic space.

Number of geodesics to the basepoint ($\arg\min Z$)



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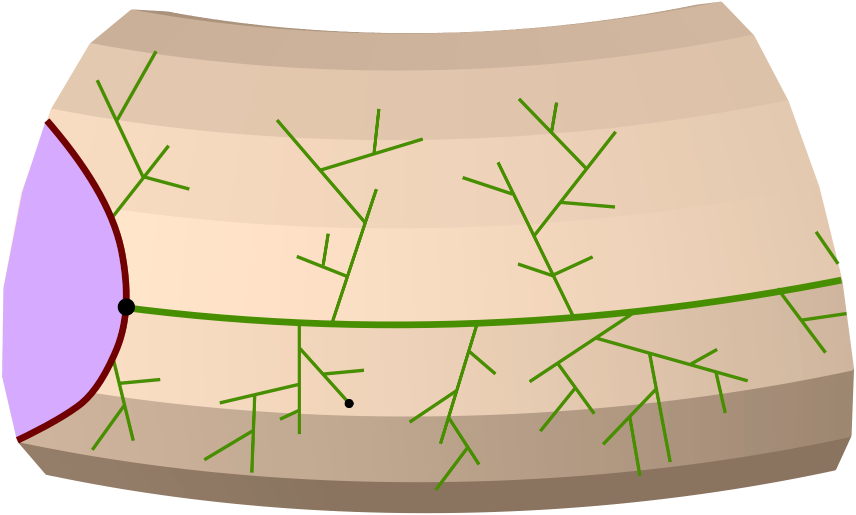

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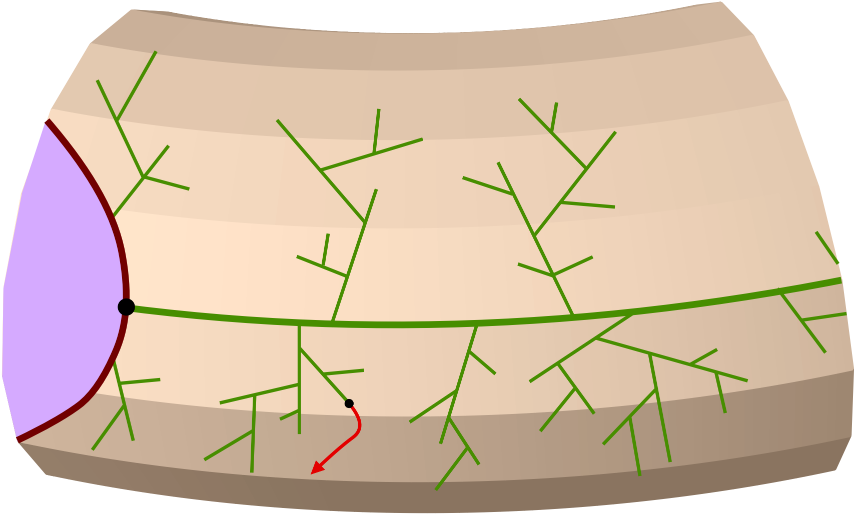

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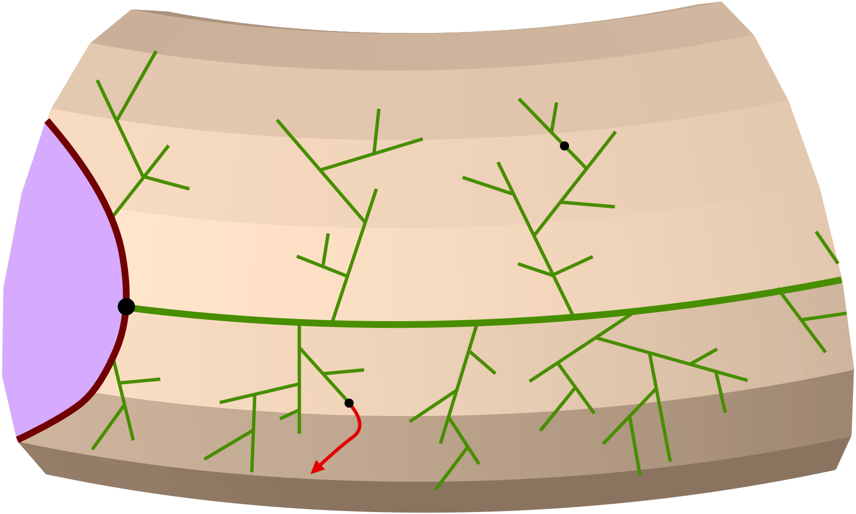

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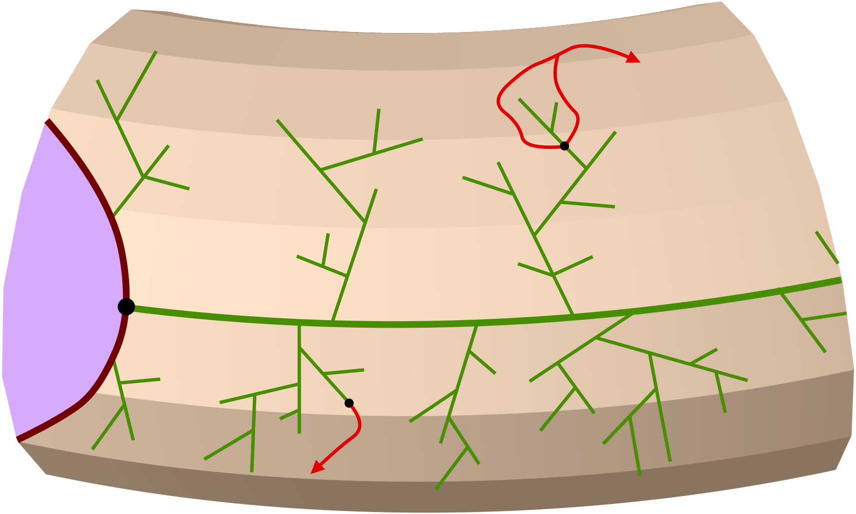

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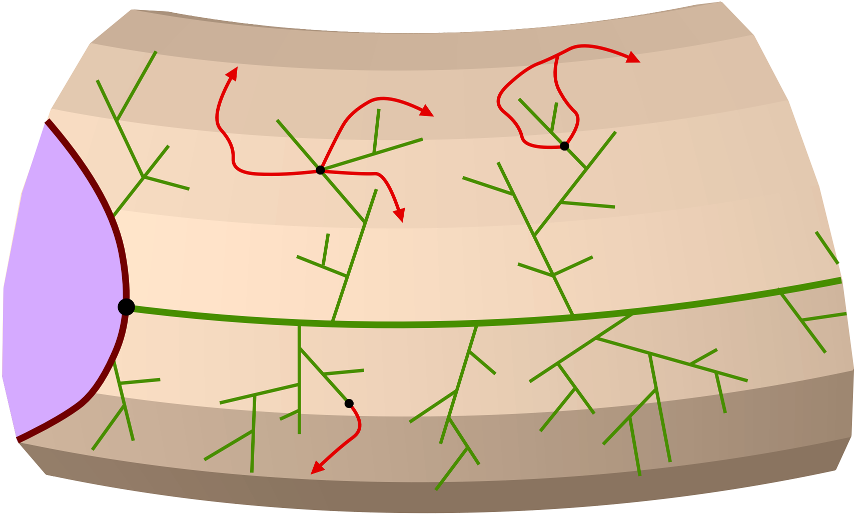

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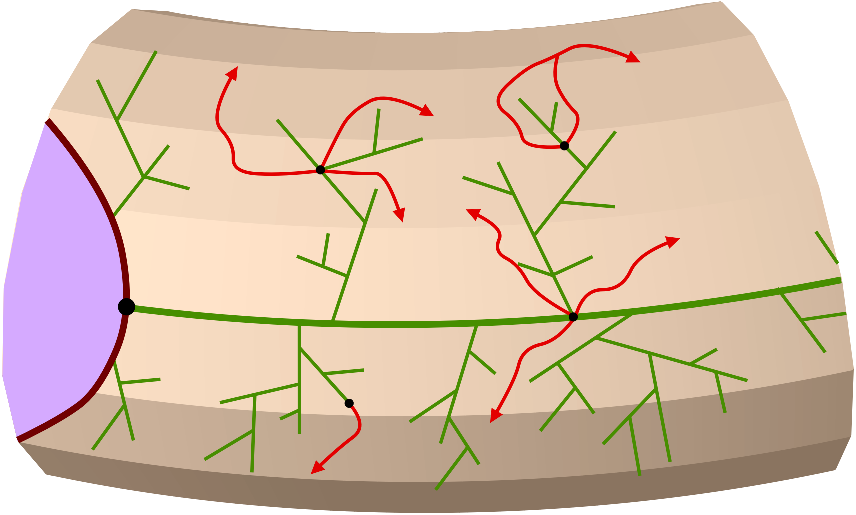

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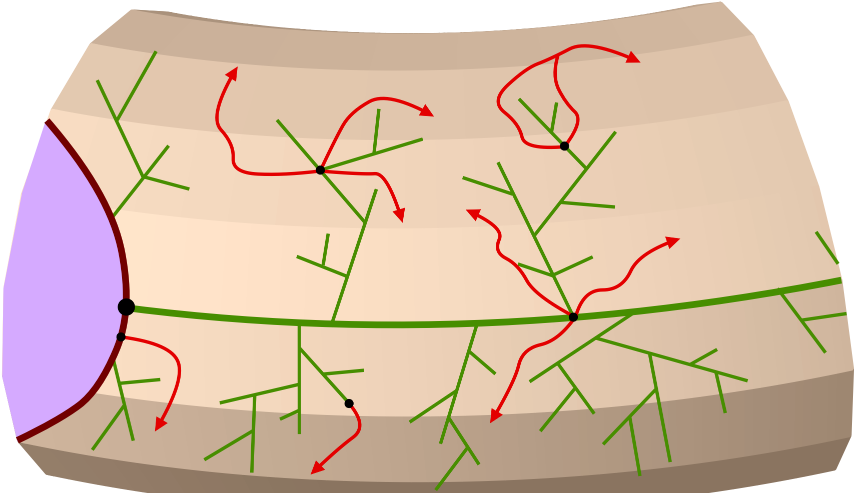

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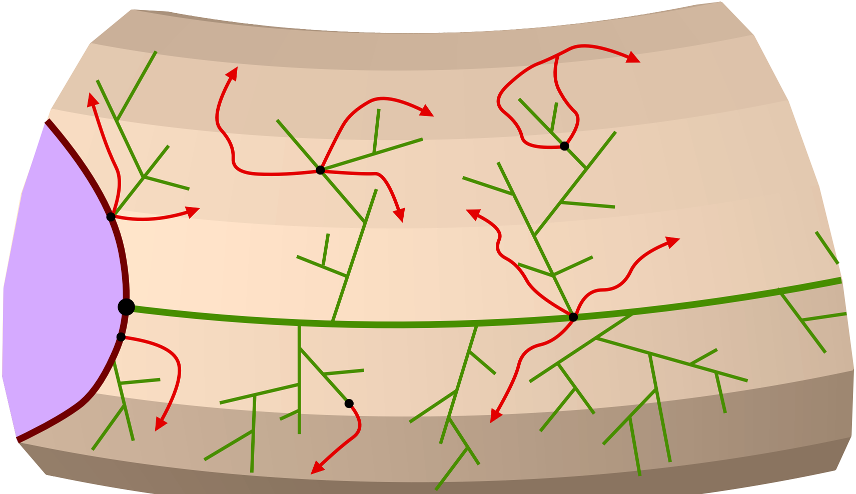
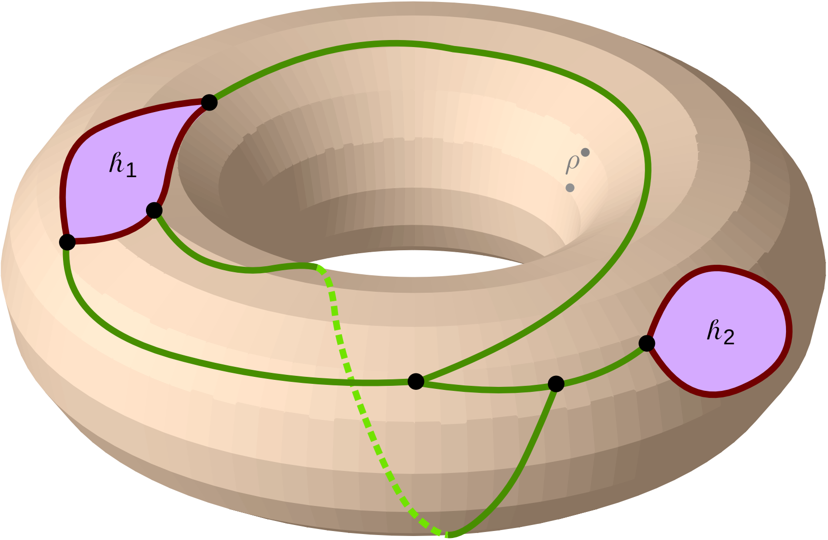


Geodesics concatenations homotopic to 0?



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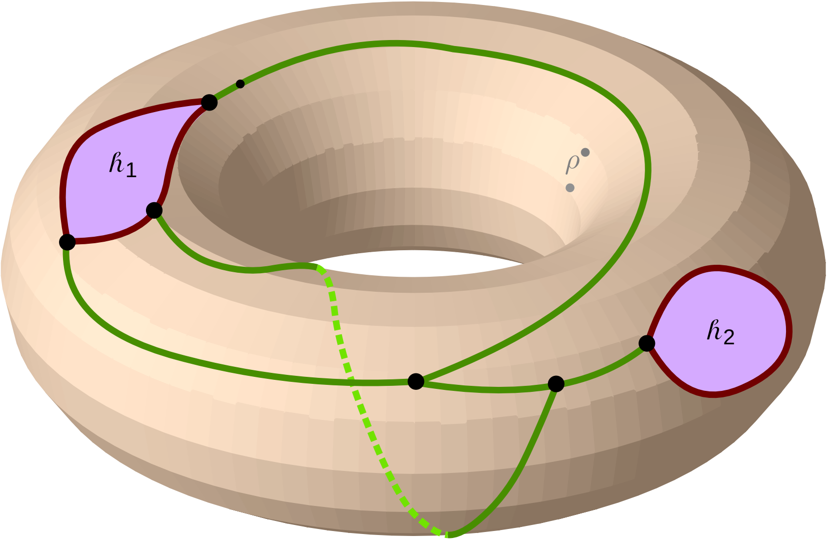

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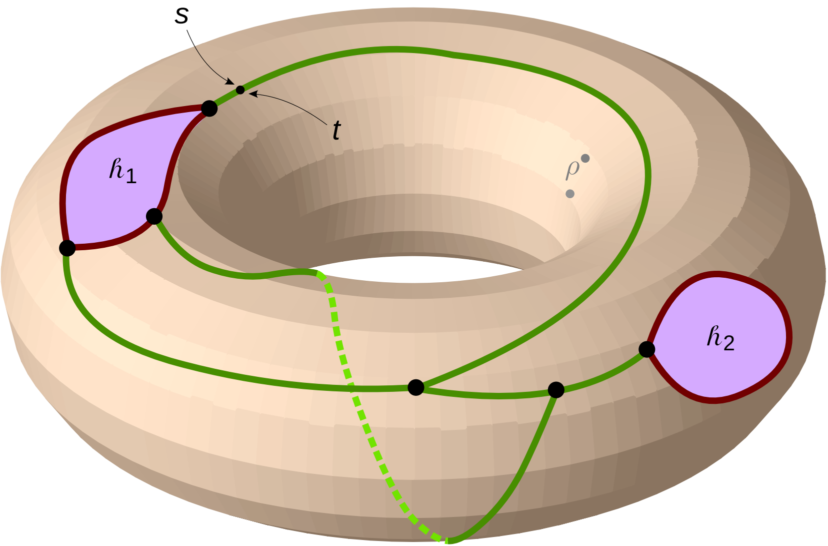

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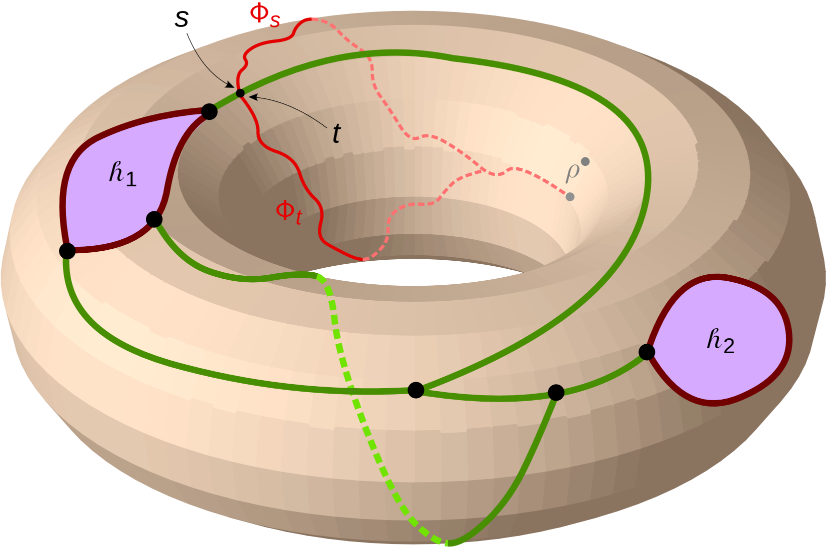

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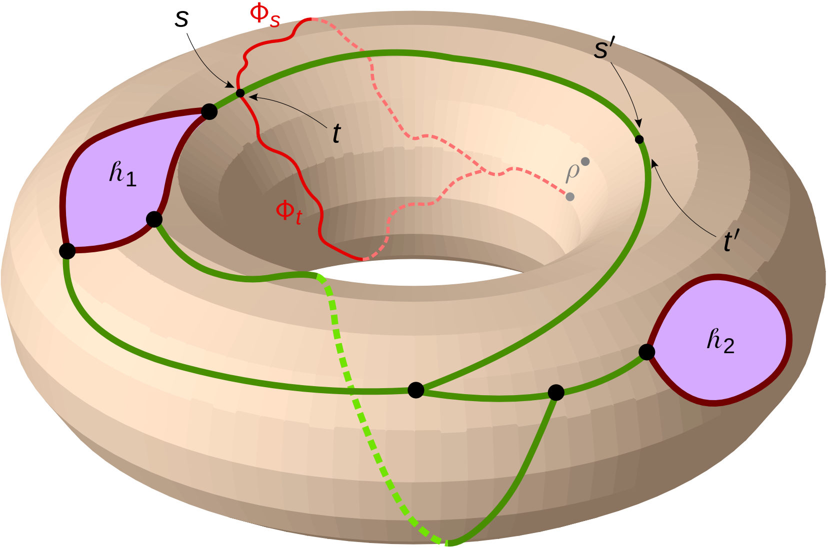

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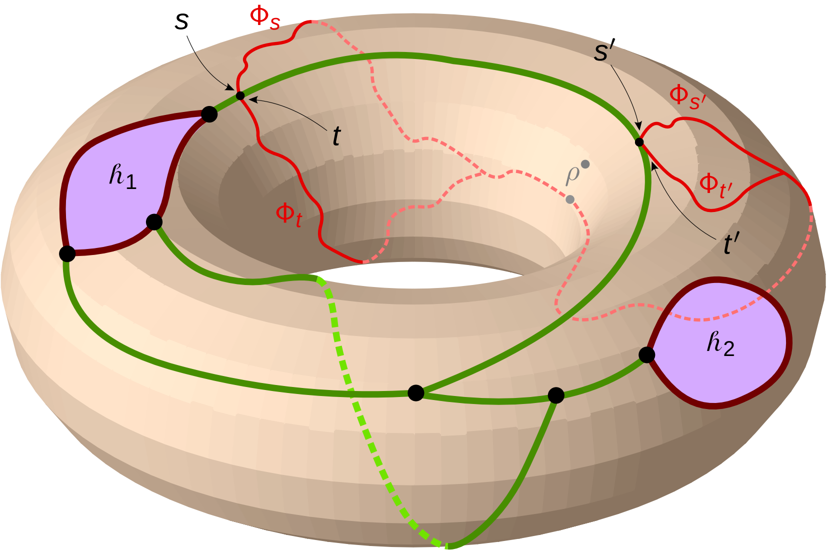
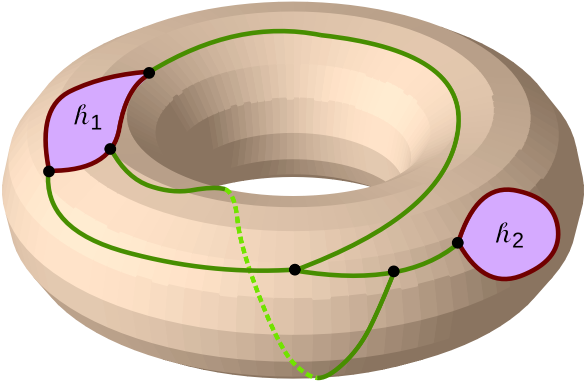
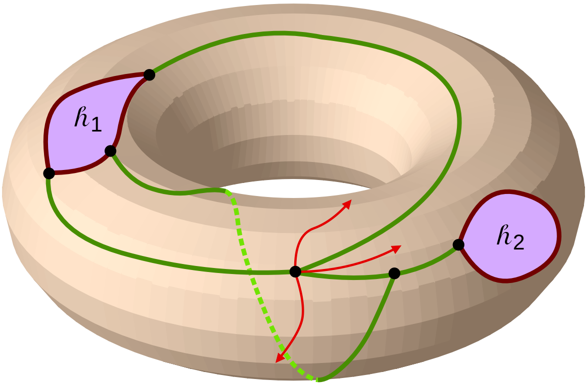


Very peculiar points



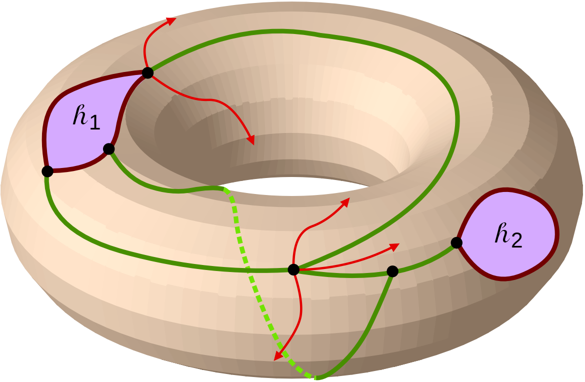
Very peculiar points

- There is a finite number of points reachable by 3 geodesics and for which every pair of geodesics make a loop not homotopic to 0.



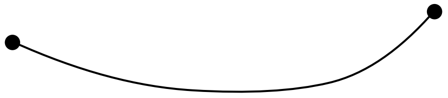
Very peculiar points

- There is a finite number of points reachable by 3 geodesics and for which every pair of geodesics make a loop not homotopic to 0.
- There is a finite number of boundary points reachable by 2 geodesics making a loop not homotopic to 0.



Confluence of geodesics

- The confluence property shown by Le Gall for the Brownian sphere easily translates for any Brownian surface.
- Basepoint: point with minimal label.



Proposition (∼ Le Gall '10)

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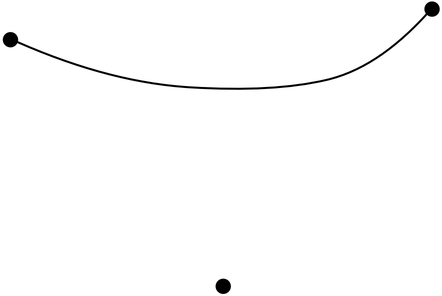

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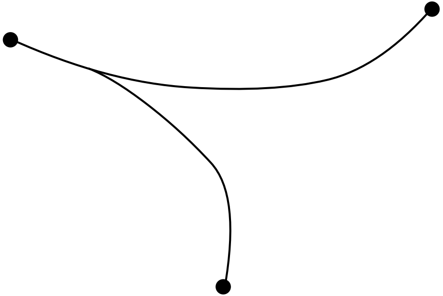

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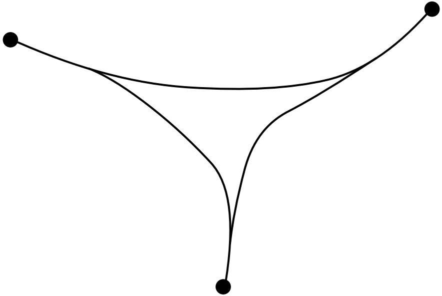

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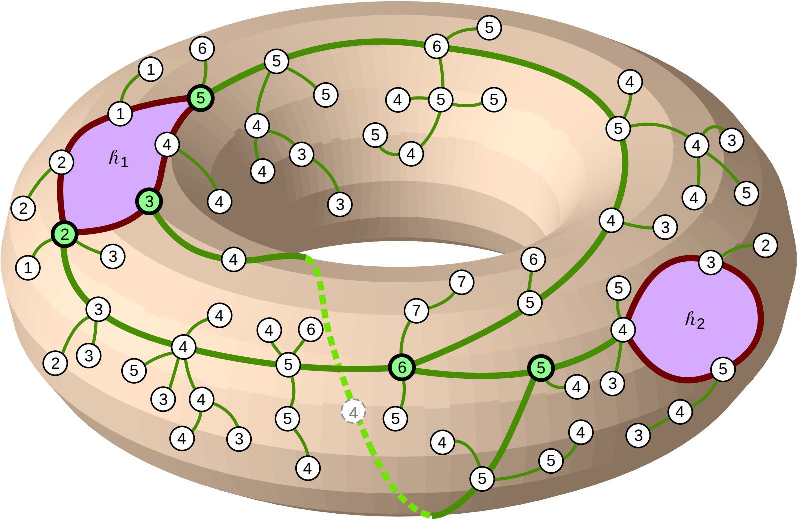

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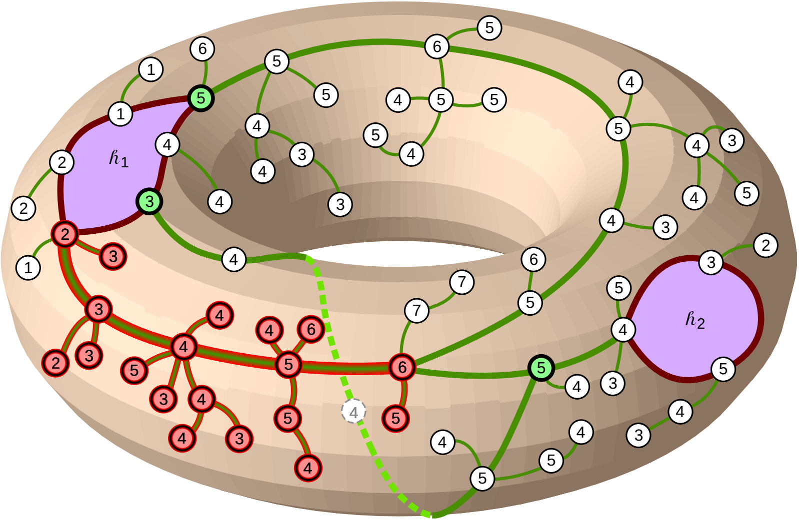

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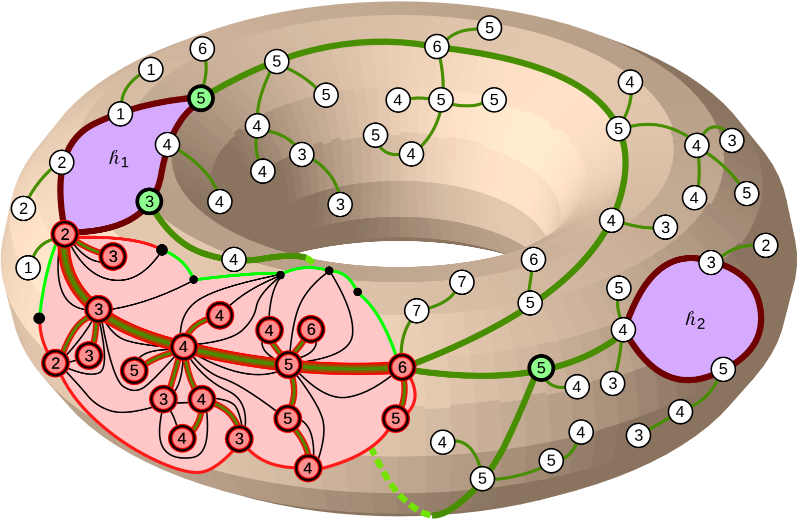

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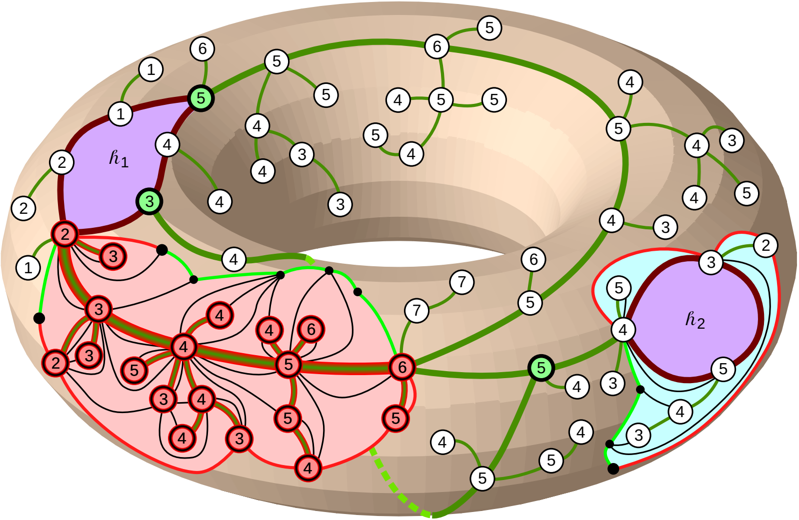

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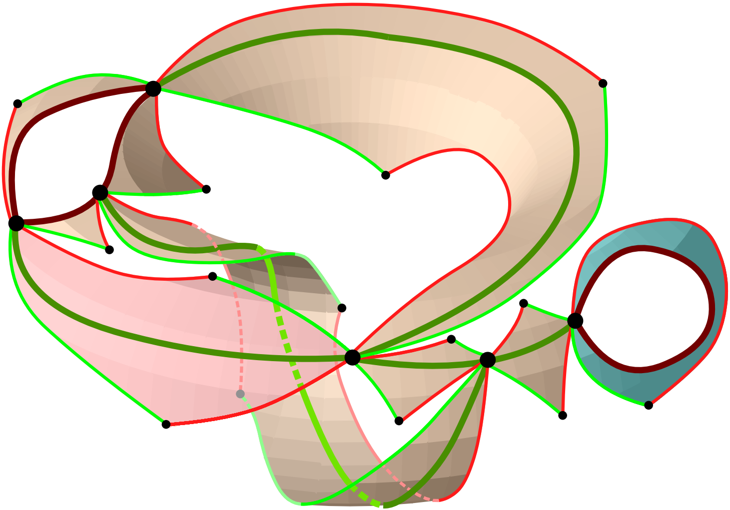

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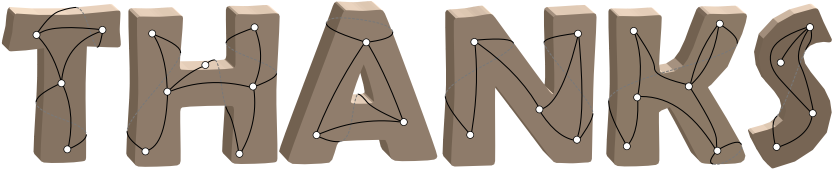
- Glue back everything together.
  - Obtain the uniqueness of the limit.
  - Obtain for free the topology and Hausdorff dimensions.


THANKS

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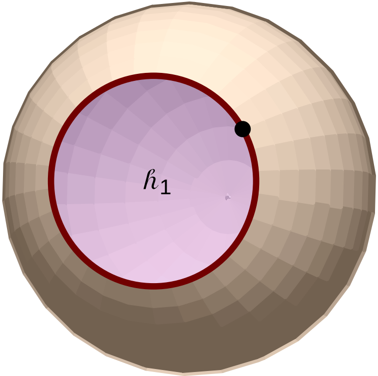

The only scheme for the disk

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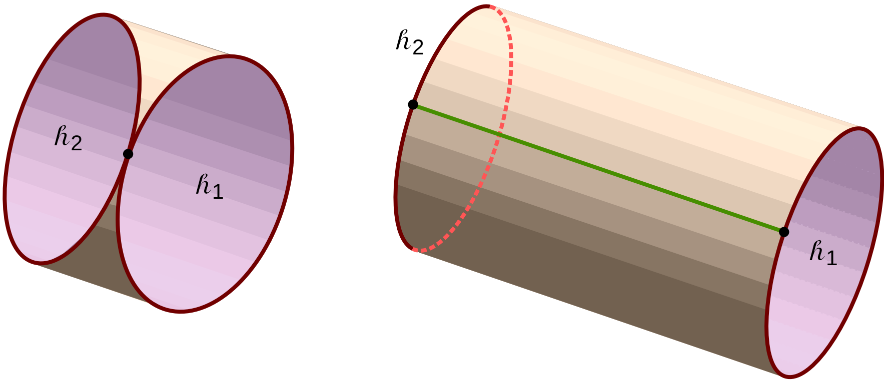

The two cylindrical schemes. Only the one on the right is dominant.

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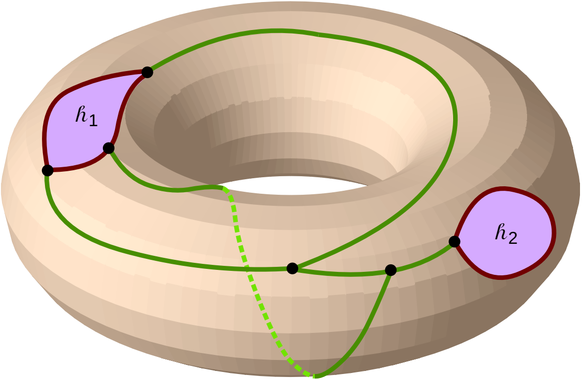

One possible (dominant) scheme for the torus with 2 holes.