Nonbijective scaling limit of maps via restriction

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joint work with Nicolas CURIEN, *Luis* FREDES, *Avelio* SEPÚLVEDA

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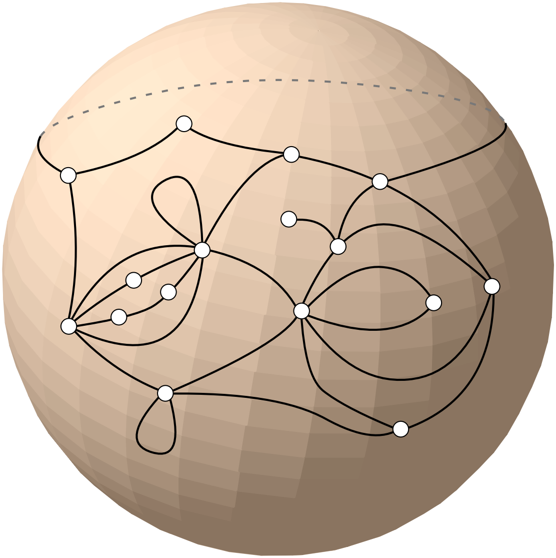
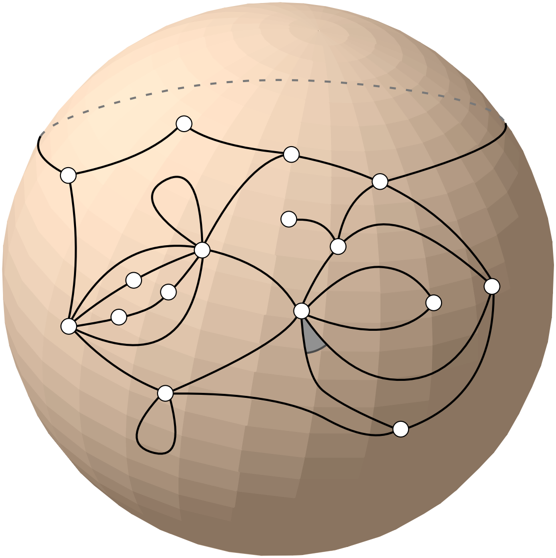
Plane maps



plane map: finite connected graph embedded in the sphere

faces: connected components of the complement

Plane maps



plane map: finite connected graph embedded in the sphere

faces: connected components of the complement

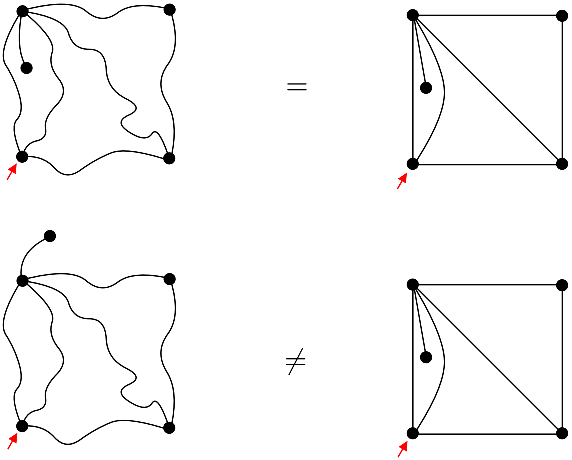
root: distinguished corner

Example of plane map

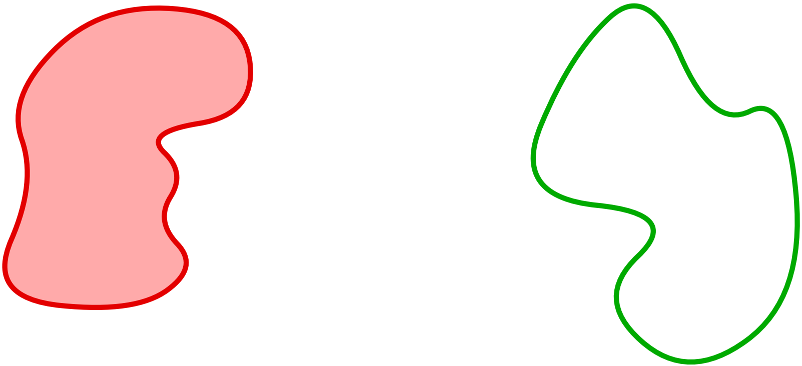
faces:
countries and
bodies of water

connected graph
no “enclaves”

Edge deformation



Gromov–Hausdorff topology: picture



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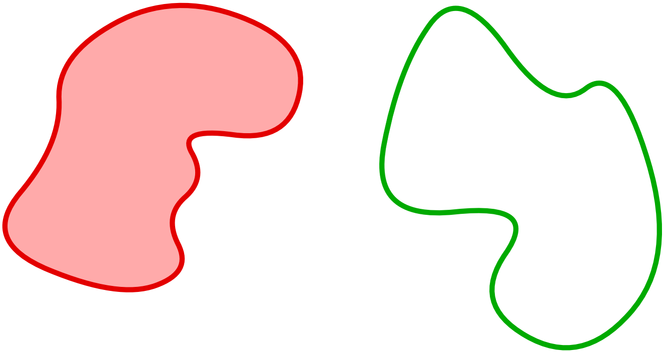

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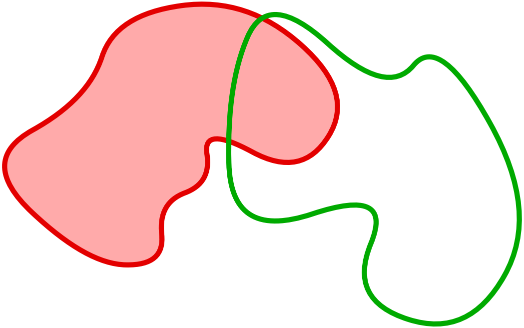

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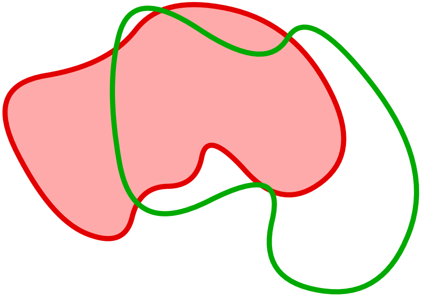

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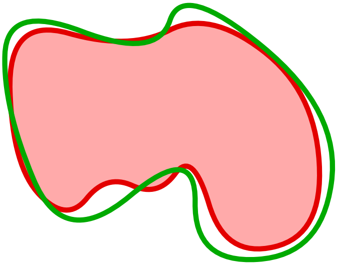

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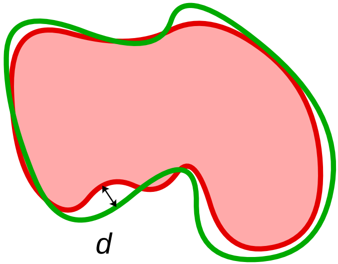

Gromov–Hausdorff topology: formal definition

- $[X, d]$: isometry class of (X, d)
- $\mathbb{M} := \{[X, d],\ (X, d) \text{ compact metric space}\}$

$$d_{\mathrm{GH}}\big([X, d], [X', d']\big) := \inf d_{\mathrm{Hausdorff}}\big(\varphi(X), \varphi'(X')\big)$$

where the infimum is taken over all metric spaces (Z, δ) and isometric embeddings $\varphi : (X, d) \to (Z, \delta)$ and $\varphi' : (X', d') \to (Z, \delta)$.

- $(\mathbb{M}, d_{\mathrm{GH}})$ is a Polish space.

Scaling limit: the Brownian sphere

- $a\,\mathbf{m}$: finite metric space obtained by endowing the vertex-set of $\mathbf{m}$ with a times the graph metric (each edge has length a).

Theorem (Le Gall '11, Miermont '11)

Let $\mathbf{q}_n$ be a uniform plane quadrangulation with n faces. The sequence $\left((8n/9)^{-1/4}\mathbf{q}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space called the Brownian sphere.

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Definition (Convergence for the Gromov–Hausdorff topology)

A sequence $(\mathcal{X}_n)$ of compact metric spaces **converges in the sense of the Gromov–Hausdorff topology** toward a metric space $\mathcal{X}$ if there exist isometric embeddings $\varphi_n : \mathcal{X}_n \to \mathcal{Z}$ and $\varphi : \mathcal{X} \to \mathcal{Z}$ into a common metric space $\mathcal{Z}$ such that $\varphi_n(\mathcal{X}_n)$ converges toward $\varphi(\mathcal{X})$ in the sense of the Hausdorff topology.

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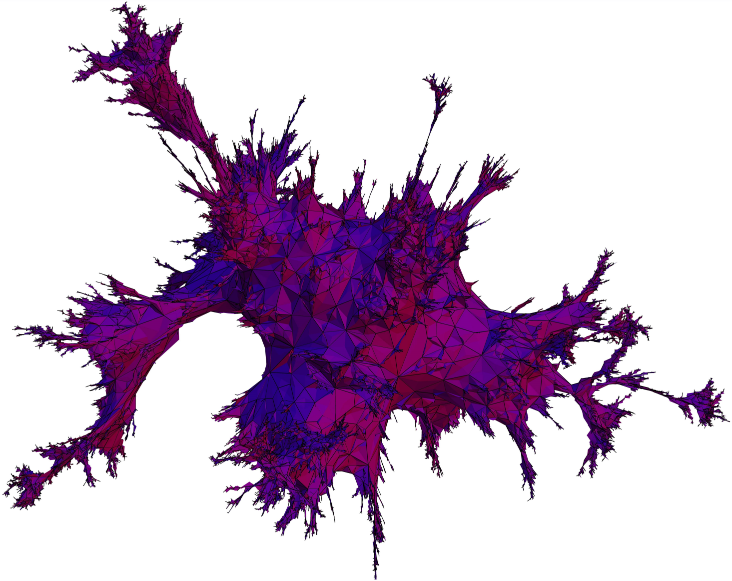
- This theorem has been proven independently by two different approaches by Miermont and by Le Gall.

Uniform plane quadrangulation with 50 000 faces



Earlier results

- Chassaing–Schaeffer ’04
  - the scaling factor is $n^{1/4}$
  - scaling limit of functionals of random uniform quadrangulations (radius, profile)
- Marckert–Mokkadem ’06
  - introduction of the Brownian sphere (called Brownian map)
- Le Gall ’07
  - the sequence of rescaled quadrangulations is relatively compact
  - any subsequential limit has the topology of the Brownian map
  - any subsequential limit has Hausdorff dimension 4
- Le Gall–Paulin ’08 & Miermont ’08
  - the topology of any subsequential limit is that of the two-sphere
- Bouttier–Guitter ’08
  - limiting joint distribution between three uniformly chosen vertices

Universality of the Brownian sphere

Many natural models of plane maps converge to the Brownian sphere (up to a model-dependent scale constant): for well-chosen maps $\mathbf{m}_n$,

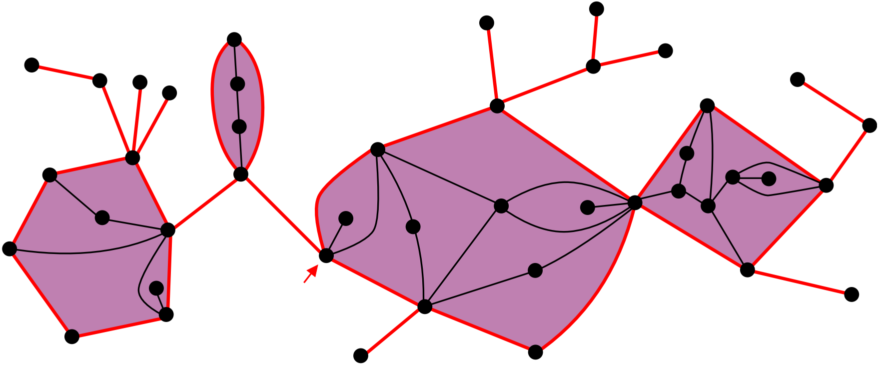
$$c\, n^{-1/4}\, \mathbf{m}_n \xrightarrow[n\to\infty]{} \text{Brownian sphere.}$$

- Le Gall '11: uniform p-angulations for $p \in \{3, 4, 6, 8, 10, \dots\}$ and Boltzmann bipartite maps with fixed number of vertices

Using Le Gall's method, many generalizations:

- Beltran and Le Gall '12: quadrangulations with no pendant edges
- Addario-Berry–Albenque '13: simple triang., simple quad.
- B.–Jacob–Miermont '14: general maps with fixed number of edges
- Abraham '14: bipartite maps with fixed number of edges
- Marzouk '17: bipartite maps with prescribed degree sequence
- Curien–Le Gall '19: triangulations with random length edges
- Addario-Berry–Albenque '20: p-angulations for odd $p \geq 5$

Plane quadrangulations with a boundary



plane quadrangulation with a boundary: plane map whose faces have degree 4, except maybe the root face

the boundary is not in general a simple curve

Scaling limit: generic case

- $\mathbf{q}_{n,p}$ uniform among quadrangulations with a boundary having area n and perimeter p
- $\ell_n/\sqrt{2n} \to L \in (0,\infty)$

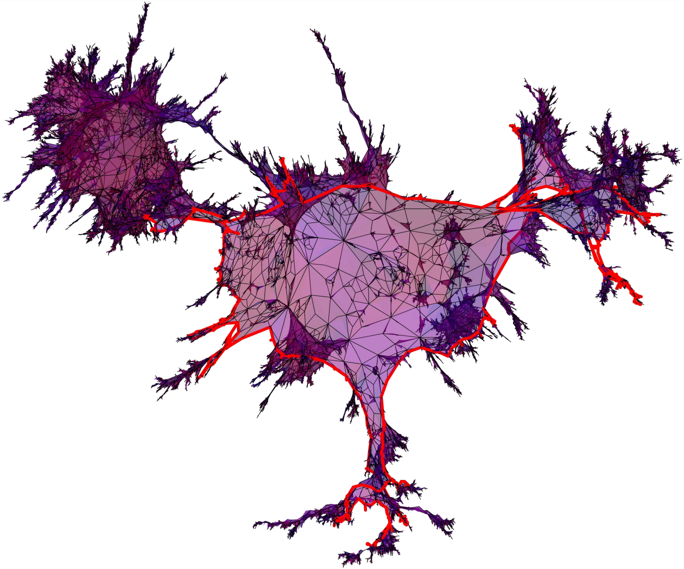
Theorem (B.–Miermont '15)

The sequence $\left((8n/9)^{-1/4}\mathbf{q}_{n,2\ell_n}\right)_{n\geq 1}$ *converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space* $\mathbf{BD}_L$ *called the Brownian disk of perimeter* L.

Theorem (B. '11)

Let $L > 0$ *be fixed. Almost surely, the space* $\mathbf{BD}_L$ *is homeomorphic to the closed unit disk of* $\mathbb{R}^2$*. Moreover, almost surely, the Hausdorff dimension of* $\mathbf{BD}_L$ *is 4, while that of its boundary* $\partial\mathbf{BD}_L$ *is 2.*

40 000 faces and boundary length 1 000



Scaling limit: degenerate cases

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The sequence $\left(\left(8n/9\right)^{-1/4}\mathbf{q}_{n,2\ell_n}\right)_{n\geq 1}$ *converges weakly in the sense of the Gromov–Hausdorff topology toward the Brownian sphere.*

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The sequence $\left((2\ell_n)^{-1/2}\mathbf{q}_{n,2\ell_n}\right)_{n\geq 1}$ *converges weakly in the sense of the Gromov–Hausdorff topology toward the Brownian Continuum Random Tree (universal scaling limit of models of random trees).*

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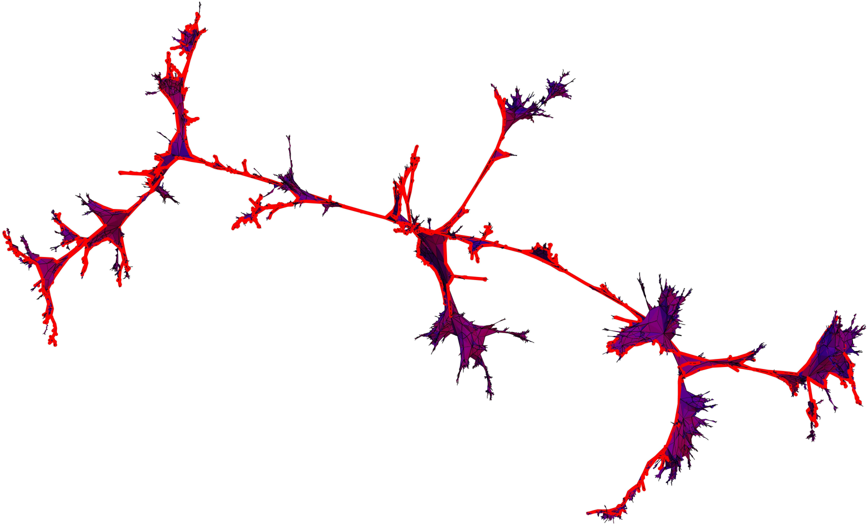
Bouttier–Guitter '09 observed these regimes in the computation of the two-point function in the same model.

10 000 faces and boundary length 2 000



Universality

Theorem (B.–Miermont '15)

Let $L \in (0, \infty)$ *be fixed,* $(\ell_n, n \geq 1)$ *be a sequence of integers such that* $\ell_n \sim L\sqrt{p(p-1)n}$ *as* $n \to \infty$*, and* $\mathbf{m}_n$ *be uniformly distributed over the set of* $2p$*-angulations with area* n *and perimeter* $2\ell_n$*. Then* $\left((4p(p-1)n/9)^{-1/4}\mathbf{m}_n\right)_{n\geq 1}$ *converges weakly in the sense of the Gromov–Hausdorff topology toward* $\mathbf{BD}_L$*.*

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Theorem (B.–Miermont '15)

Let $\mathbf{m}_n$ be a uniform random bipartite map with area n and perimeter $2\ell_n$, where $\ell_n \sim 3L\sqrt{n/2}$ for some $L > 0$. Then $\left((2n)^{-1/4}\mathbf{m}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward $\mathbf{BD}_L$.

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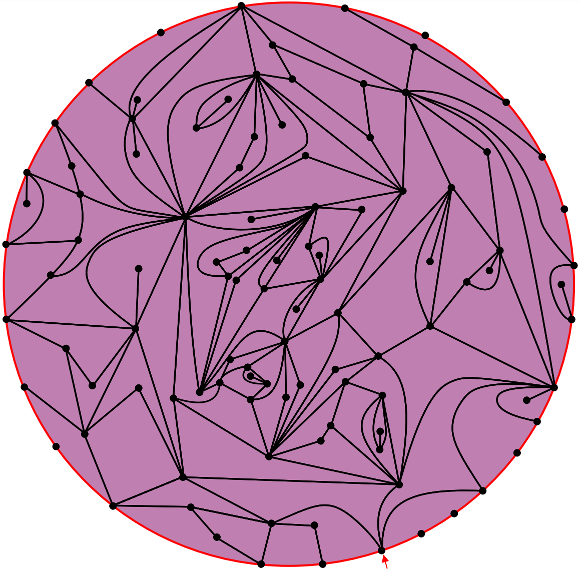
- More universality results for bipartite Boltzmann maps conditionned on their number of vertices, faces or edges.

Plane quadrangulations with a **simple** boundary



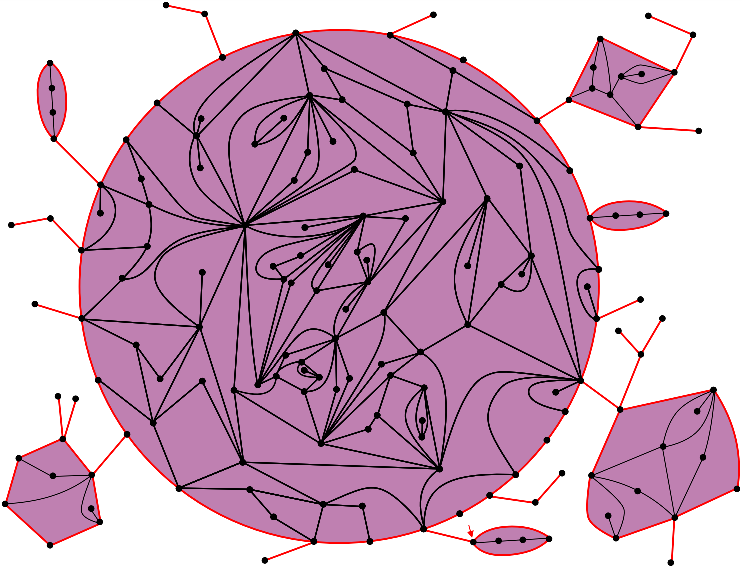
Scaling limit: generic case

- $\tilde{\mathbf{q}}_{n,p}$ uniform among quadrangulations with a simple boundary having area n and perimeter p
- $\ell_n/\sqrt{2n} \to L \in (0,\infty)$

Theorem (B.–Curien–Fredes–Sepúlveda '21)

The sequence $\left((8n/9)^{-1/4}\tilde{\mathbf{q}}_{n,2\ell_n}\right)_{n\geq 1}$ *converges weakly in the sense of the Gromov–Hausdorff topology toward* $\mathbf{BD}_{3L}$*, the Brownian disk of perimeter* $3L$*.*

Core of a **pointed** map



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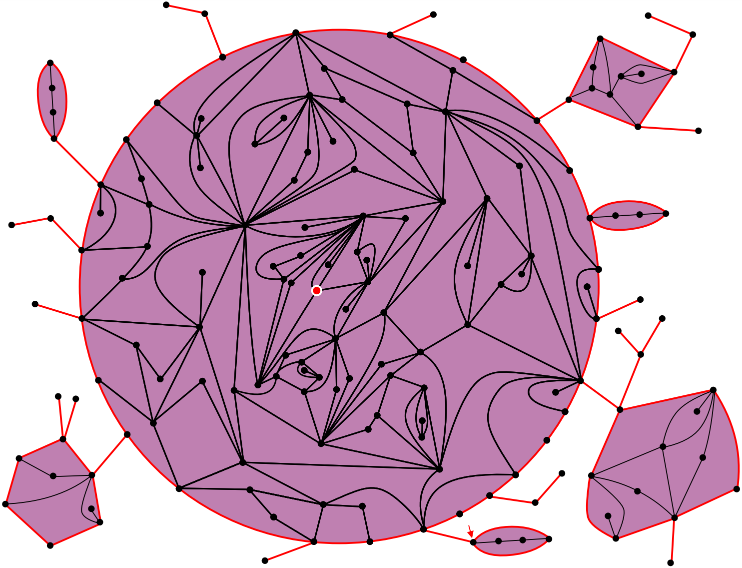

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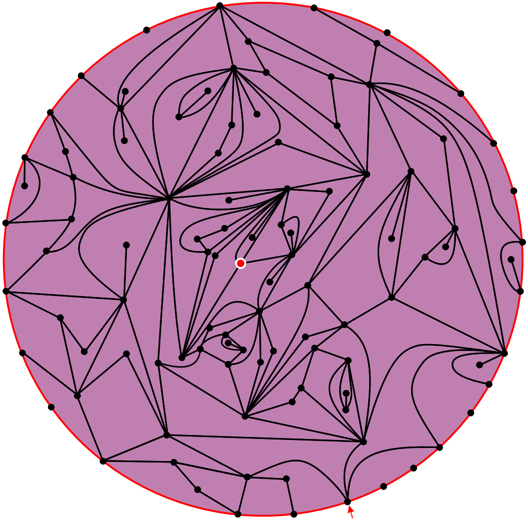

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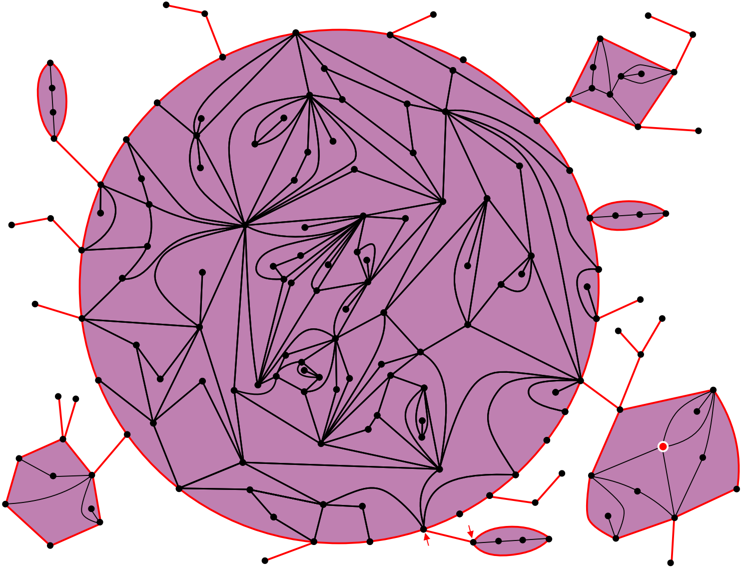


Core of a **pointed** map

cemetery point $\wp$

Asymptotics

- $\mathbf{q}^\bullet_{n,p}$ uniform among pointed quadrangulations with a boundary having area n and perimeter p
- $\ell_n/\sqrt{2n} \to L \in (0,\infty)$

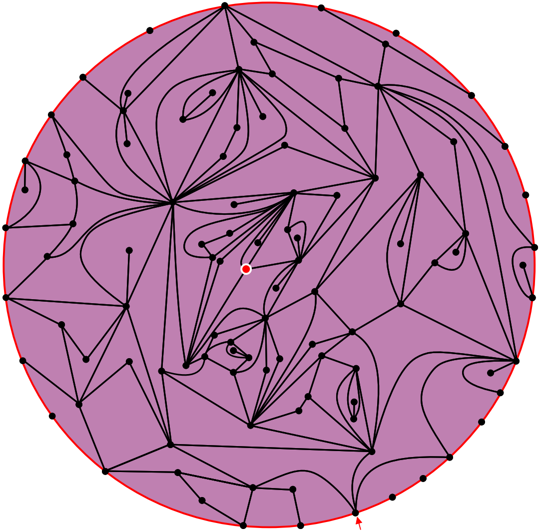
Proposition (Gwynne–Miller '19)

- $\mathbb{P}\Big(\mathrm{Core}\left(\mathbf{q}^\bullet_{n,6\ell_n}\right) \neq \wp\Big) \to 1$ *as* $n \to \infty$
- $\dfrac{\left\|\mathrm{Core}\left(\mathbf{q}^\bullet_{n,6\ell_n}\right)\right\|}{n} \xrightarrow[n\to\infty]{(\mathbb{P})} 1$ *area*
- $\dfrac{\left|\partial\,\mathrm{Core}\left(\mathbf{q}^\bullet_{n,6\ell_n}\right)\right|}{2\ell_n} \xrightarrow[n\to\infty]{(\mathbb{P})} 1$ *perimeter*

Consequences

- Conditionally given $A_n = \|\operatorname{Core}(\mathbf{q}^\bullet_{n,6\ell_n})\|$ and $P_n = |\partial \operatorname{Core}(\mathbf{q}^\bullet_{n,6\ell_n})|$, provided that $A_n > n/2$, the r.v. $\operatorname{Core}(\mathbf{q}^\bullet_{n,6\ell_n})$ is uniform among quadrangulations with a **simple** boundary having area A_n and perimeter P_n.
- $\left(\left(\frac{9}{8n}\right)^{1/4} \mathbf{q}^\bullet_{n,6\ell_n}, \left(\frac{9}{8n}\right)^{1/4} \operatorname{Core}\left(\mathbf{q}^\bullet_{n,6\ell_n}\right)\right) \xrightarrow[n\to\infty]{(d)} \left(\mathbf{BD}_{3L}, \mathbf{BD}_{3L}\right)$.
- $\tilde{\mathbf{q}}^\bullet_{n,2\ell_n} \approx \tilde{\mathbf{q}}^\bullet_{A_n,P_n} \sim \operatorname{Core}\left(\mathbf{q}^\bullet_{n,6\ell_n}\right)$.
- Lift to a conditional convergence when A_n and P_n are fixed.
- Prove that the distributions of “large parts” of $\operatorname{Core}\left(\mathbf{q}^\bullet_{n,6\ell_n}\right)$ and of $\tilde{\mathbf{q}}^\bullet_{n,2\ell_n}$ may be rendered arbitrary close in total variation distance.

Restrictions



- $\mathbf{q}^\bullet$, $n \in \mathbb{N}$, $\varepsilon > 0$

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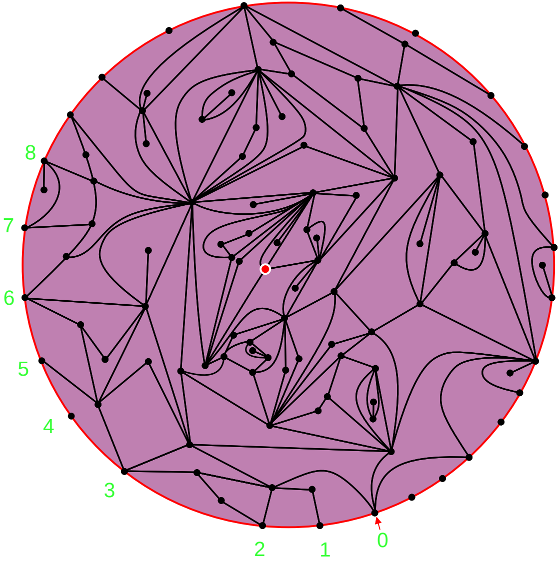

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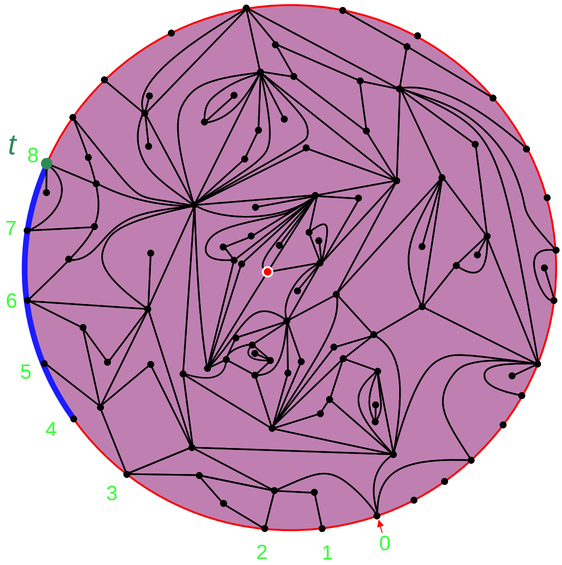
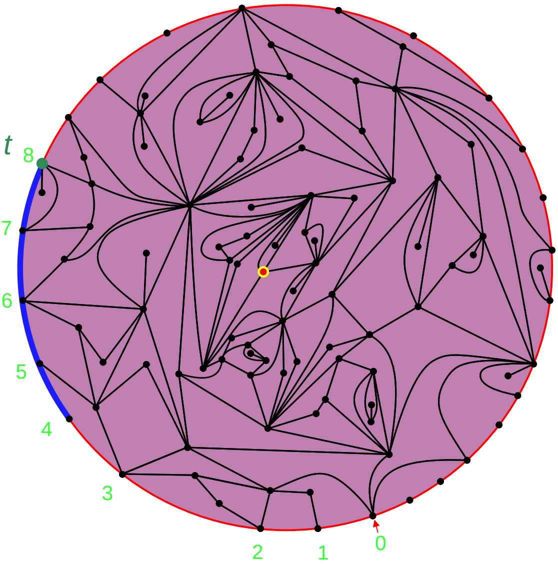


- $\mathbf{q}^\bullet$, $n \in \mathbb{N}$, $\varepsilon > 0$
- bdry vertices from $\left\lfloor \left(\frac{1}{3} - \varepsilon\right) 2\ell_n \right\rfloor$ to $\left\lfloor \frac{2\ell_n}{3} \right\rfloor$

Restrictions



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- grow balls

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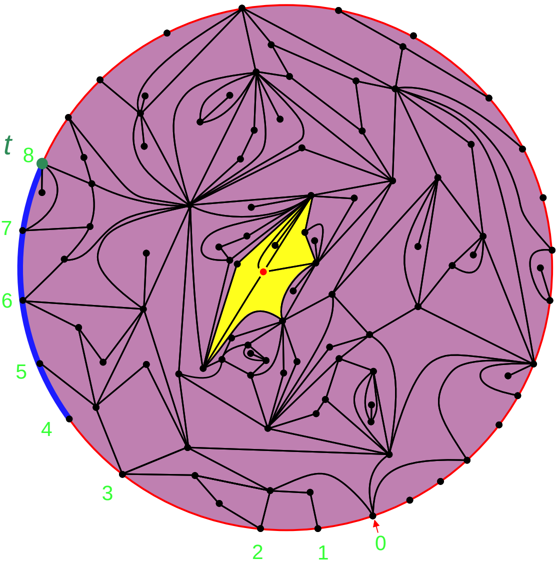

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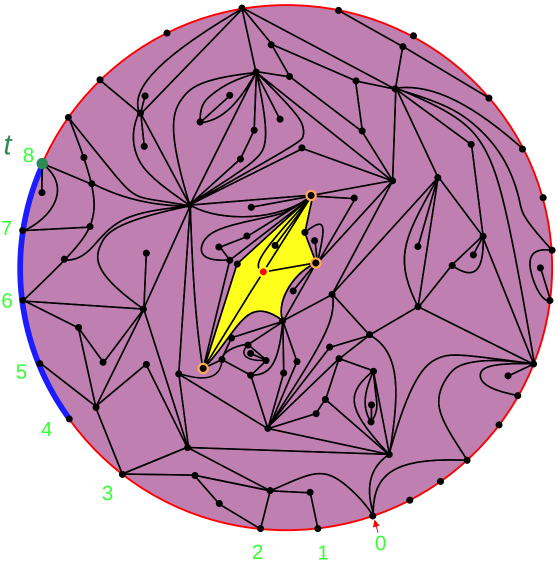

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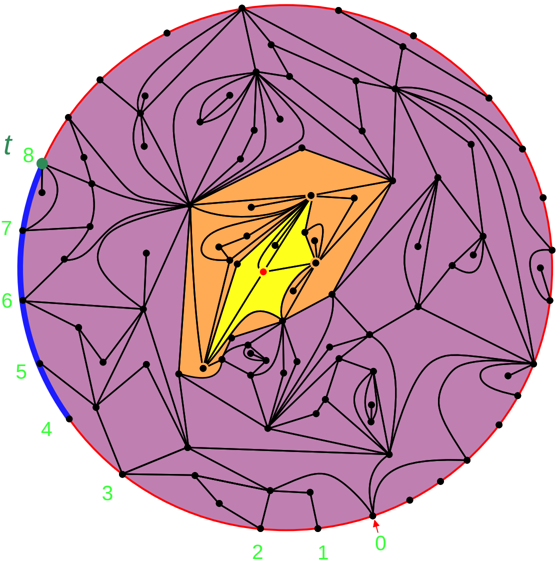

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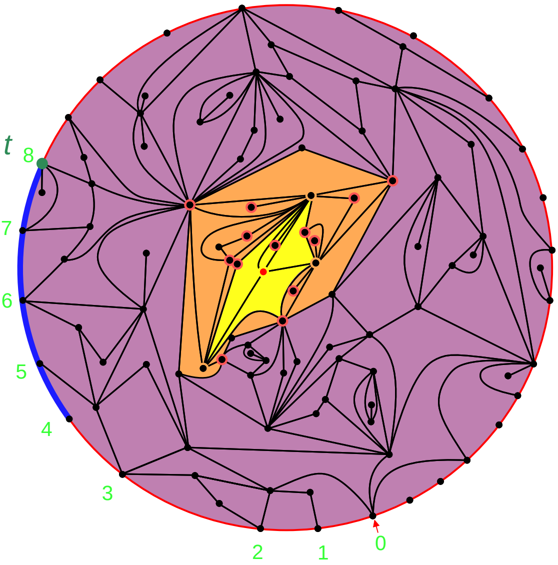

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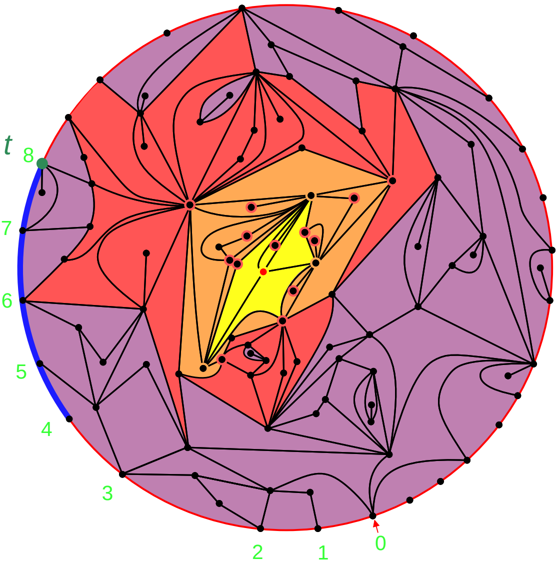

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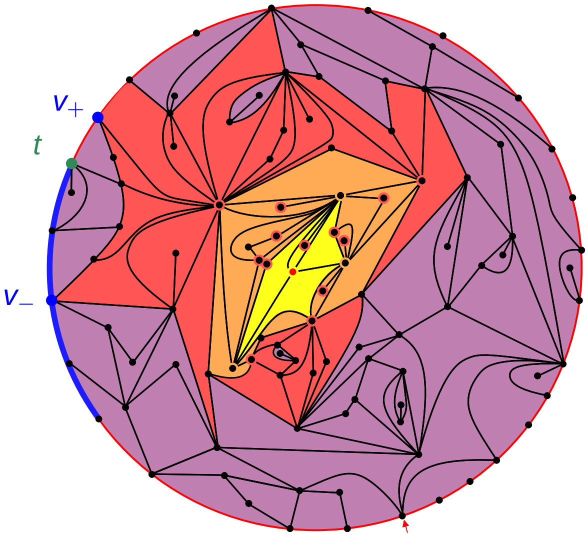
- grow balls
- stop when hitting —

Restrictions



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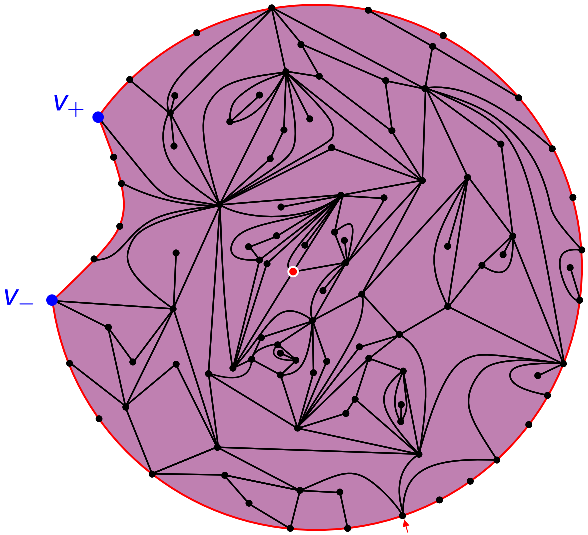

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- stop when hitting —
- hull $\mathcal{R}_n^\varepsilon(\mathbf{q}^\bullet)$

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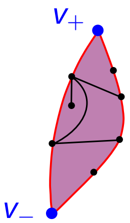

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- grow balls
- stop when hitting —
- hull $\mathcal{R}_n^\varepsilon(\mathbf{q}^\bullet)$
- complement $\bar{\mathcal{R}}_n^\varepsilon(\mathbf{q}^\bullet)$

Technical estimates

- $\underbrace{X_n := \tilde{\mathbf{q}}^\bullet_{n,2\ell_n}}_{\textit{model under study}}$ $\underbrace{Y_n := \mathrm{Core}\left(\mathbf{q}^\bullet_{n,6\ell_n}\right)}_{\textit{reference model}}$ $a_n := \left(\frac{9}{8n}\right)^{1/4}$
- $a_n Y_n \xrightarrow[n\to\infty]{(d)} Y := \mathbf{BD}_{3L}$.

Technical estimates

- $\underbrace{\textcolor{red}{X_n} := \tilde{\mathbf{q}}^\bullet_{n,2\ell_n}}_{\textit{model under study}}$ $\qquad \underbrace{\textcolor{green}{Y_n} := \mathrm{Core}\left(\mathbf{q}^\bullet_{n,6\ell_n}\right)}_{\textit{reference model}}$ $\qquad a_n := \left(\frac{9}{8n}\right)^{1/4}$
- $a_n \textcolor{green}{Y_n} \xrightarrow[n\to\infty]{(d)} \textcolor{green}{Y} := \mathbf{BD}_{3L}$.

Proposition (Restrictions are close)

For $\varepsilon > 0$, $\quad \lim_{n\to\infty} \mathrm{d_{TV}}\big(\mathcal{R}^\varepsilon_n(\textcolor{red}{X_n}), \mathcal{R}^\varepsilon_n(\textcolor{green}{Y_n})\big) = 0.$

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Proposition (Leftover is small)

- *For* $\delta > 0$, $\quad \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\Big(\mathrm{d_{GH}}\left(a_n Y_n, a_n \mathcal{R}^{\varepsilon}_n(Y_n)\right) > \delta\Big) = 0$.
- *For* $\delta > 0$, $\quad \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\Big(\mathrm{d_{GH}}\left(a_n X_n, a_n \mathcal{R}^{\varepsilon}_n(X_n)\right) > \delta\Big) = 0$.

Proof

- Skorohod's embedding theorem: $a_n Y_n \to Y$ a.s.
- Fix a bounded uniformly continuous real-valued function f, $\eta > 0$.
- $\delta > 0$ such that $d_{GH}(\mathcal{X}, \mathcal{Y}) < 3\delta \implies |f(\mathcal{X}) - f(\mathcal{Y})| < \eta$.

$$\Big|\mathbb{E}\big[f(a_n X_n) - f(Y)\big]\Big| \le \mathbb{E}\big[|f(a_n X_n) - f(Y)|\,;\, d_{GH}(Y, a_n X_n) < 3\delta\big] + \mathbb{E}\big[|f(a_n X_n) - f(Y)|\,;\, d_{GH}(Y, a_n X_n) \ge 3\delta\big]$$

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$$+ \mathbb{E}\big[|f(a_n X_n) - f(Y)|\,;\, \mathrm{d_{GH}}(Y, a_n X_n) \geq 3\delta\big]$$

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$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty} = 0$$

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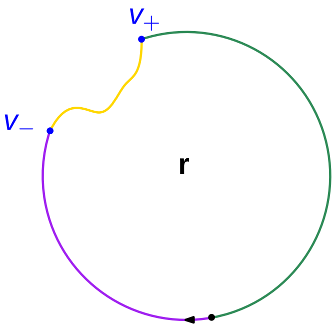
maximal coupling theorem for $\varepsilon > 0$ fixed

Closeness of restrictions via counting

Proposition (Restrictions are close)

For $\varepsilon > 0$, $\lim_{n\to\infty} \mathrm{d_{TV}}\big(\mathcal{R}^\varepsilon_n(X_n), \mathcal{R}^\varepsilon_n(Y_n)\big) = 0.$

- $\mathbb{P}\big(\mathcal{R}^\varepsilon_n(\tilde{\mathbf{q}}^\bullet_{n',p'}) = \mathbf{r}\big)$ explicit quotient of numbers of quadrangulations with a simple boundary having given area and perimeter.
- Explicit formula for these numbers [Bouttier–Guitter '09].
- $\dfrac{\|Y_n\|}{n} \to 1, \quad \dfrac{|\partial Y_n|}{2\ell_n} \to 1$ in probability.



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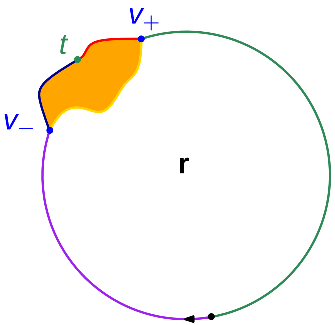


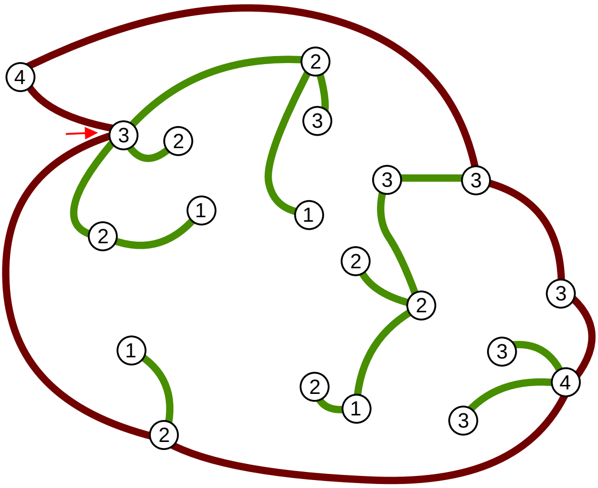
Goal

$$Y_n := \mathrm{Core}\left(\mathbf{q}^\bullet_{n,6\ell_n}\right)$$

Proposition (Leftover is small in reference model)

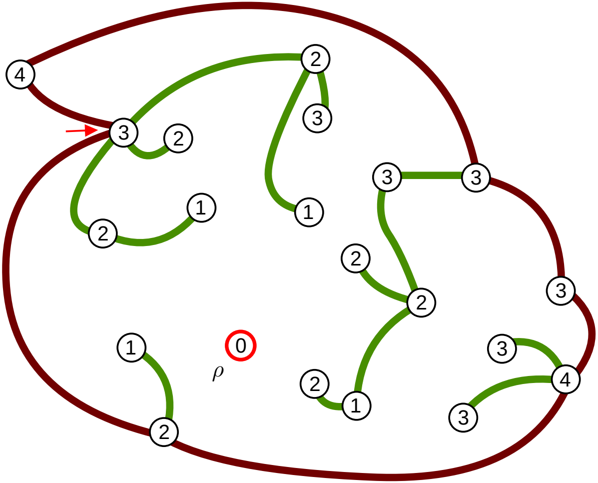
For $\delta > 0$, $\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\mathbb{P}\Big(\mathrm{d_{GH}}\big(a_n Y_n, a_n\mathcal{R}_n^\varepsilon(Y_n)\big) > \delta\Big) = 0.$

Encoding a quadrangulation with general boundary



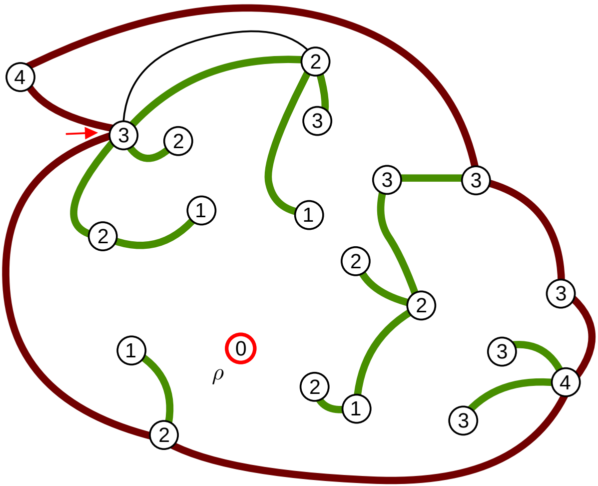
- Take a labeled forest.

Encoding a quadrangulation with general boundary



- Take a labeled forest.
- Add a vertex ρ inside the unique face.

Encoding a quadrangulation with general boundary



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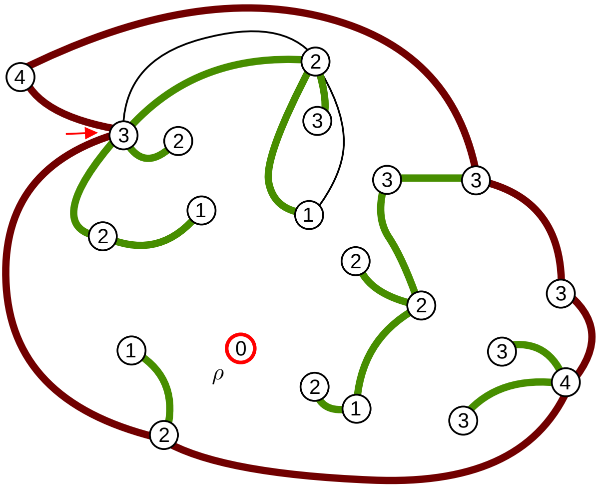
- Link every corner to the first subsequent corner having a strictly smaller label.

Encoding a quadrangulation with general boundary



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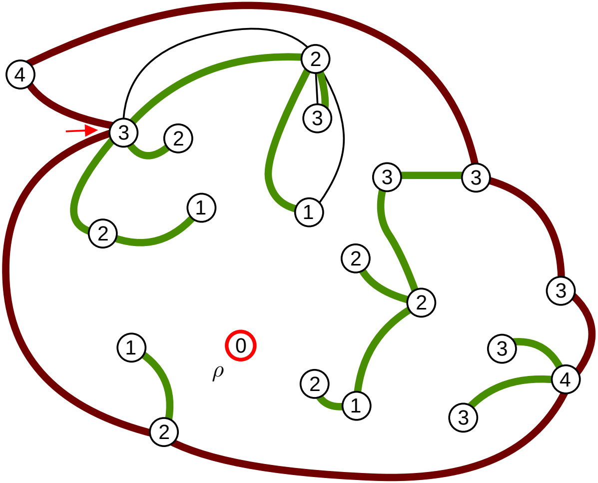

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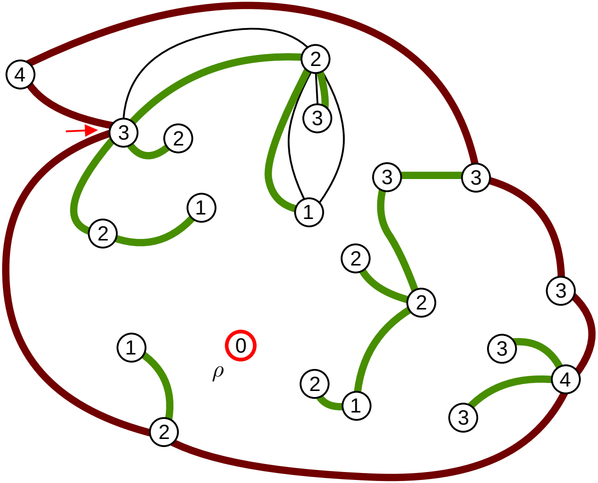

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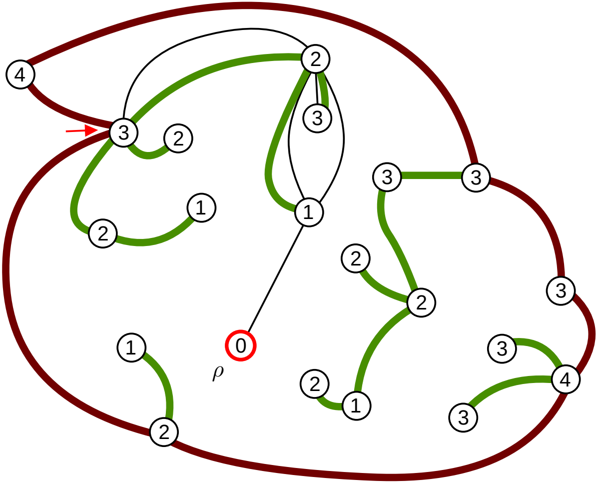

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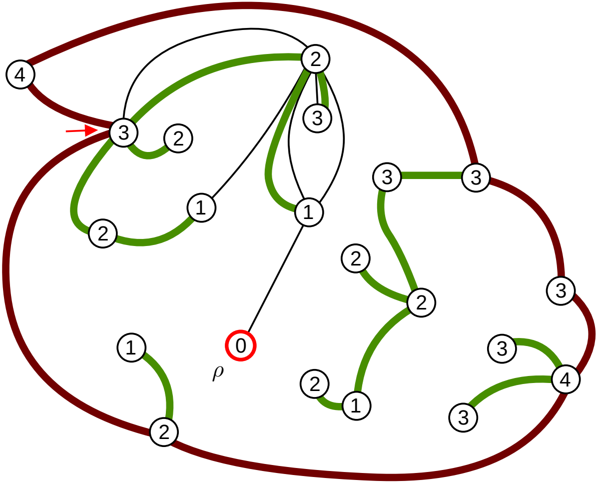

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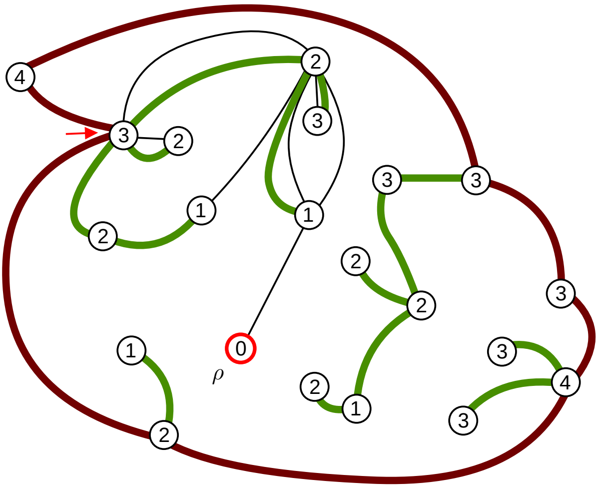

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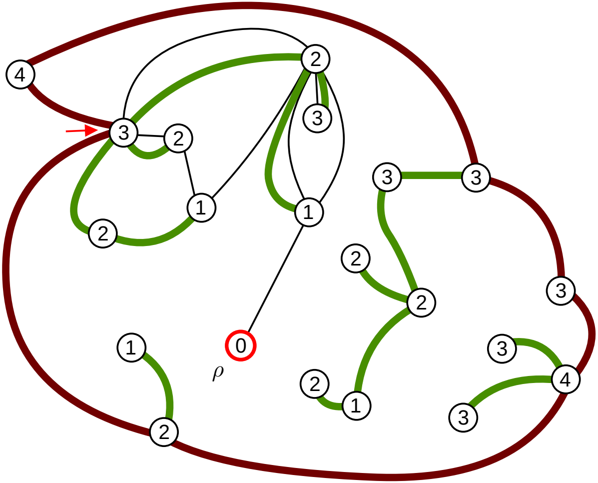

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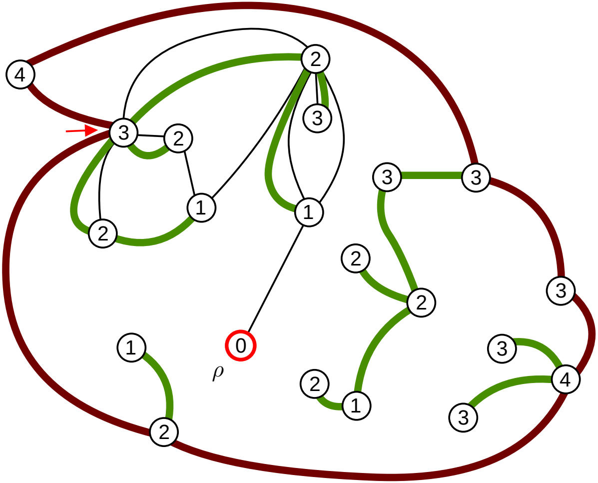

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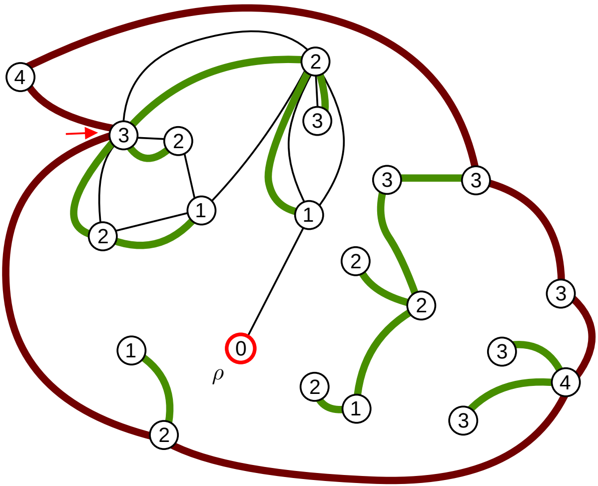

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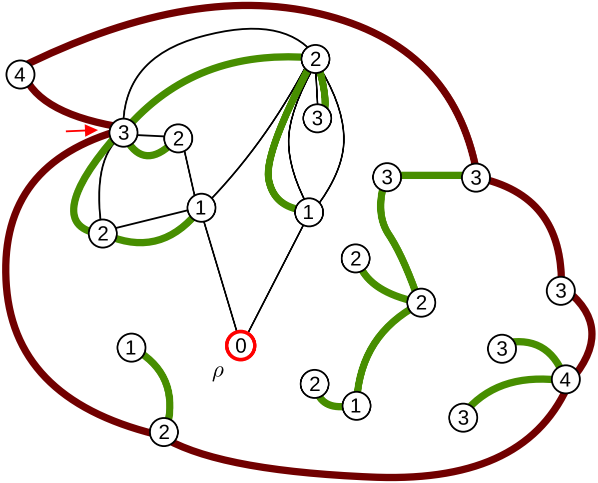

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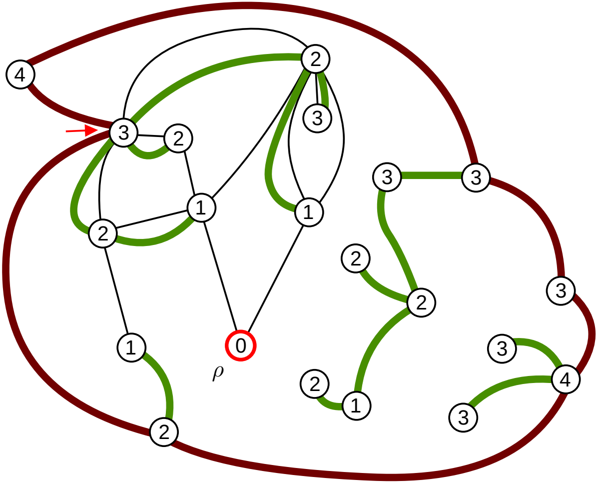

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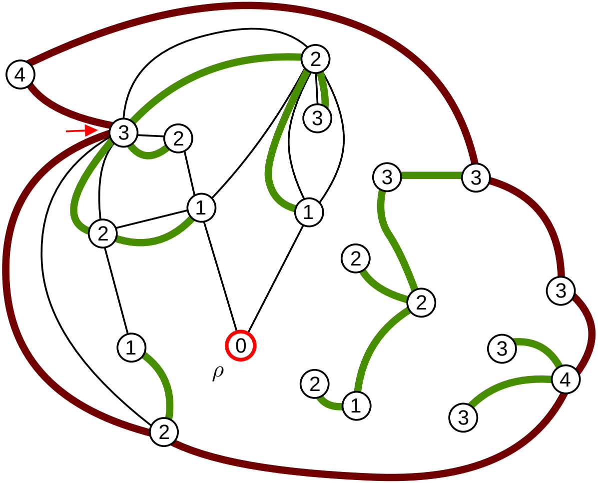

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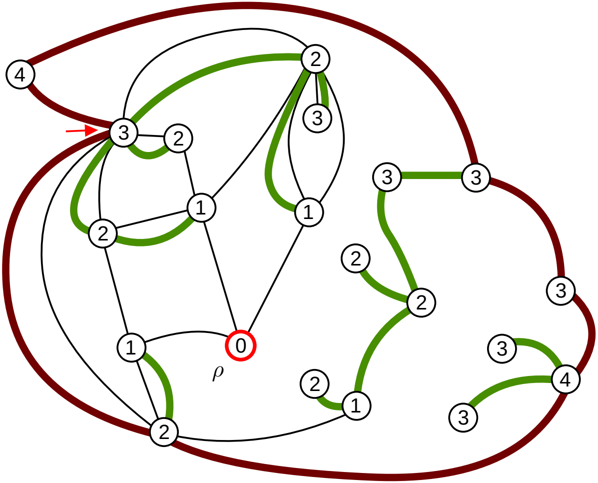

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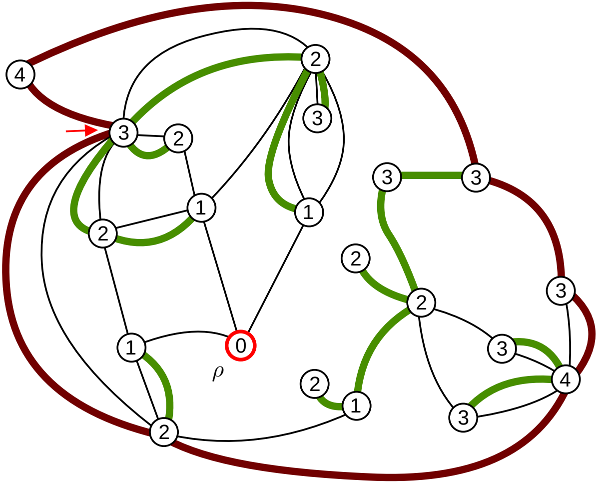

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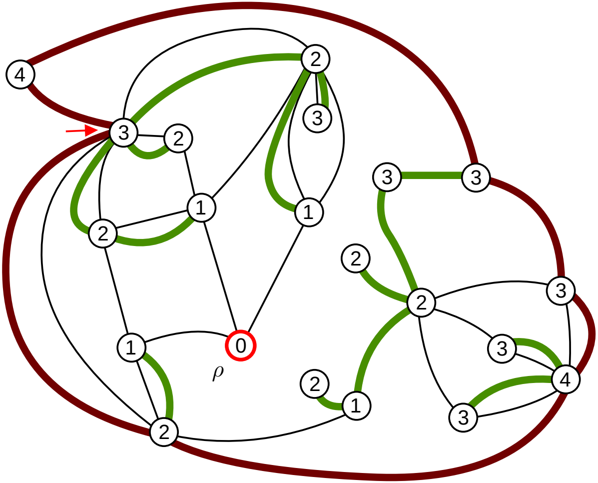

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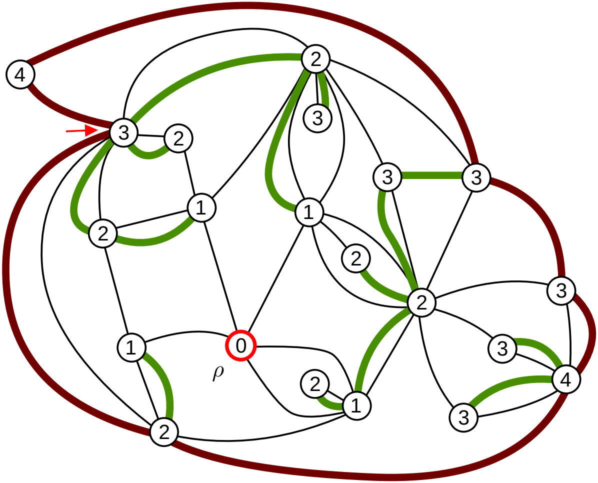

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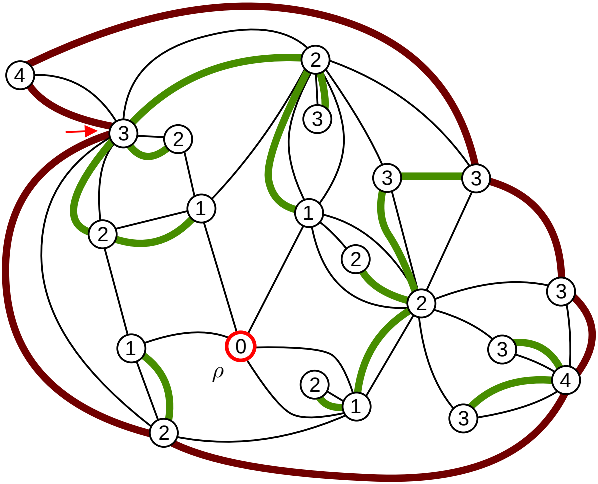

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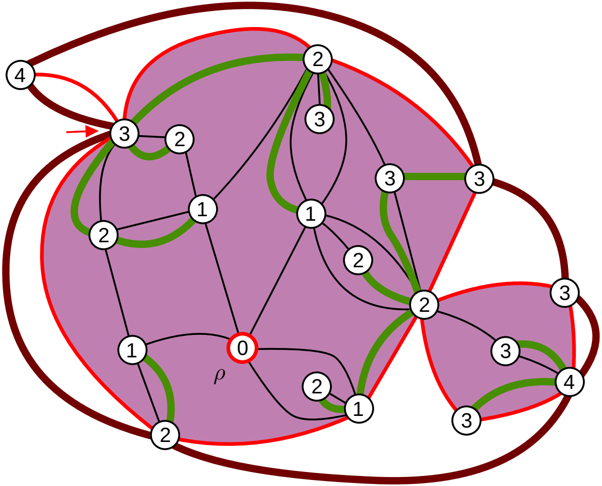

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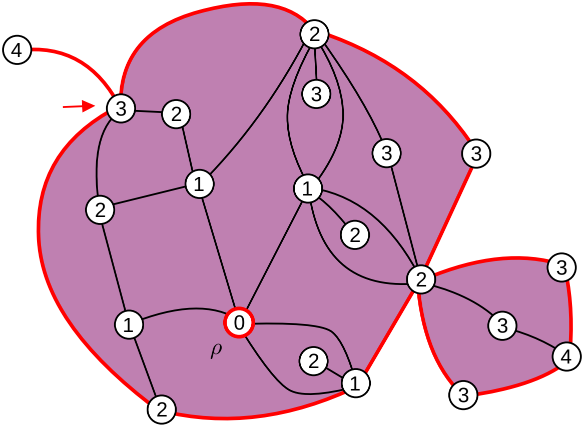

- Take a labeled forest.
- Add a vertex ρ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.
- Remove the initial edges.

Key facts

Theorem (Bouttier–Di Francesco–Guitter (generalization of Cori–Vauquelin–Schaeffer))

The previous construction yields a bijection between the following:

- *labeled forests with n edges and l trees;*
- *pointed quadrangulations with a boundary having n internal faces and boundary length $2l$ such that the root vertex is farther away from the distinguished vertex than the previous vertex in clockwise order around the boundary.*

Lemma

The labels of the forest become the distances in the map to the distinguished vertex ρ.

Scaling limit results

- Up to a shift, after scaling,
  - the tree root labels converge to a Brownian bridge $(B_t)_{0\le t\le 1}$,
  - the labeled trees converge to Brownian trees with Brownian labels.

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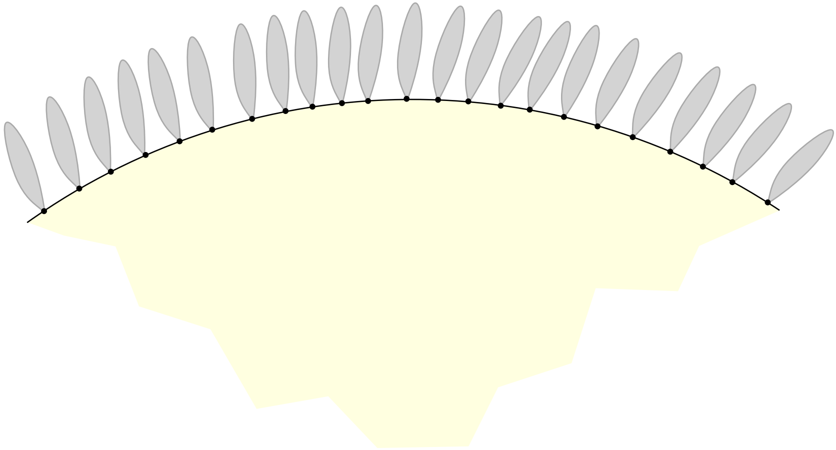
- Up to a shift, after scaling,
  - the tree root labels converge to a Brownian bridge $(B_t)_{0\le t\le 1}$,
  - the labeled trees converge to Brownian trees with Brownian labels.
- The boundary of the core is "uniformly spread" along the boundary.
  - Arrange the boundary vertices in contour order.
  - Let J_k be the first index of the k-th vertex of the core.
  - Then, $\left(\dfrac{J_{\lfloor 2\ell_n t\rfloor}}{6\ell_n}\right)_{0\le t\le 1} \xrightarrow[n\to\infty]{(d)} (t)_{0\le t\le 1}$.

Entrapping the complement of the restriction



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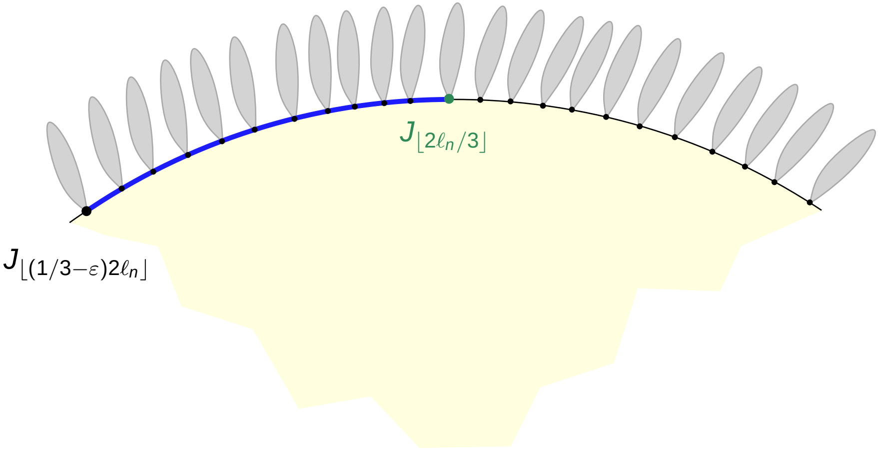

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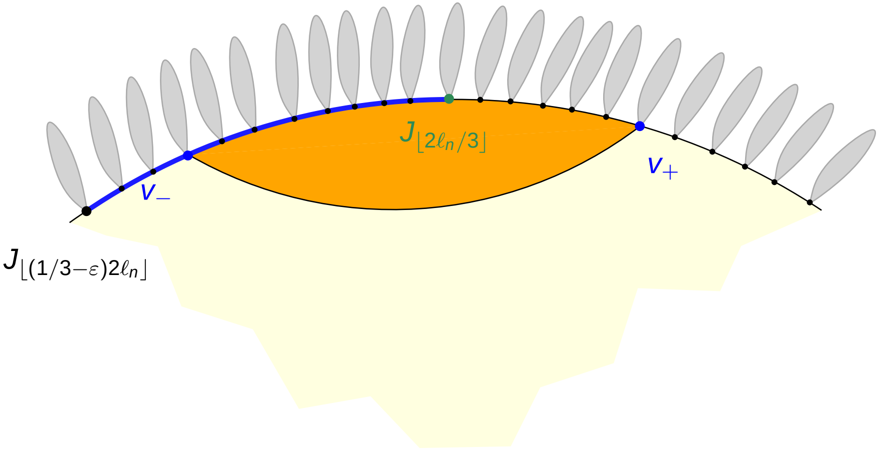

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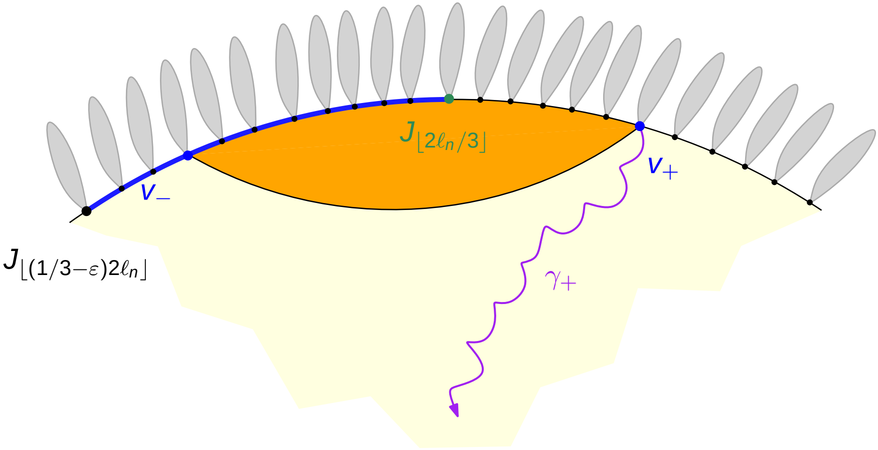

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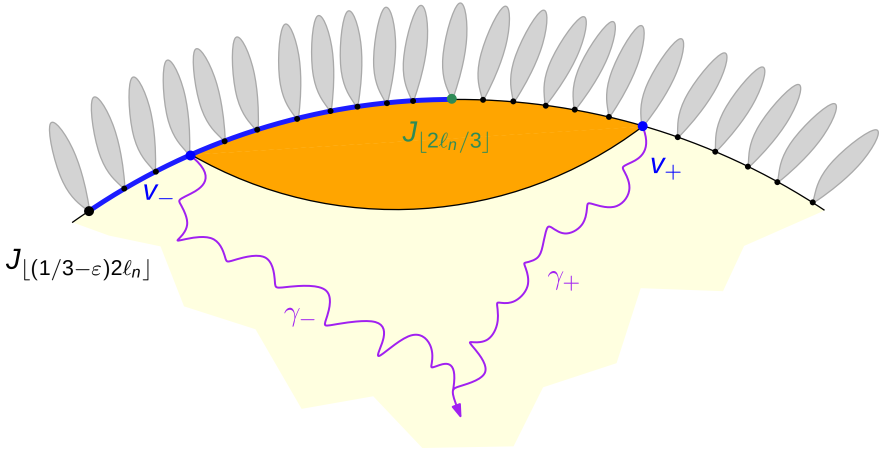
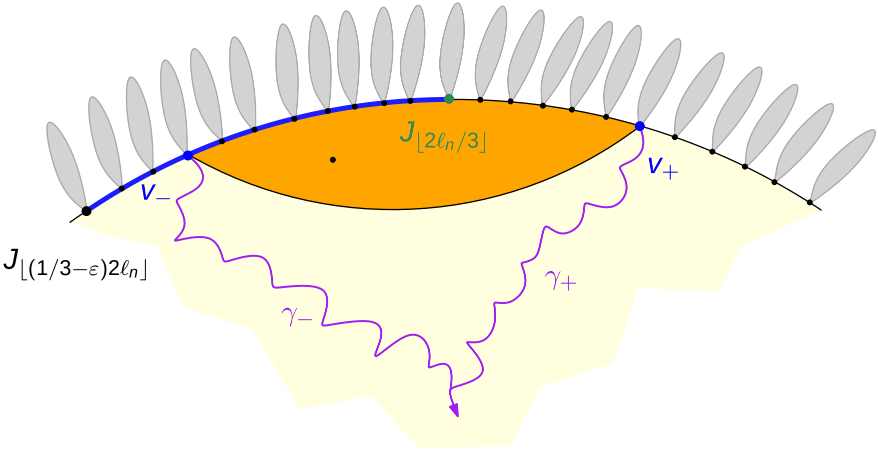
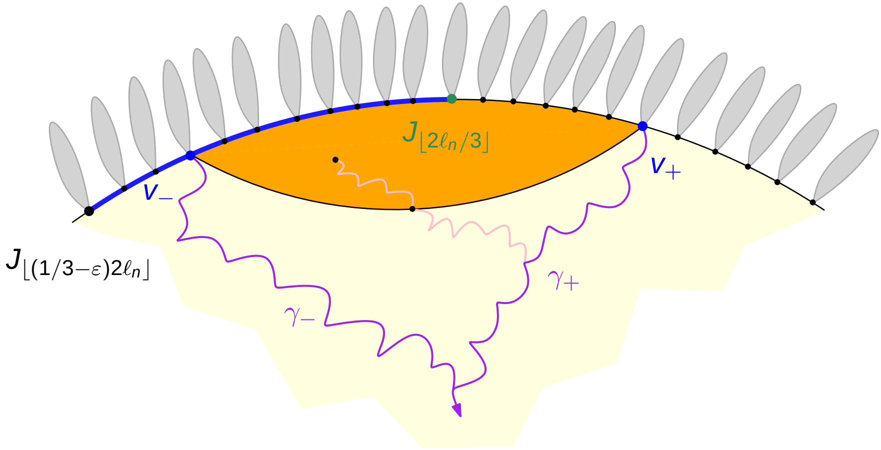


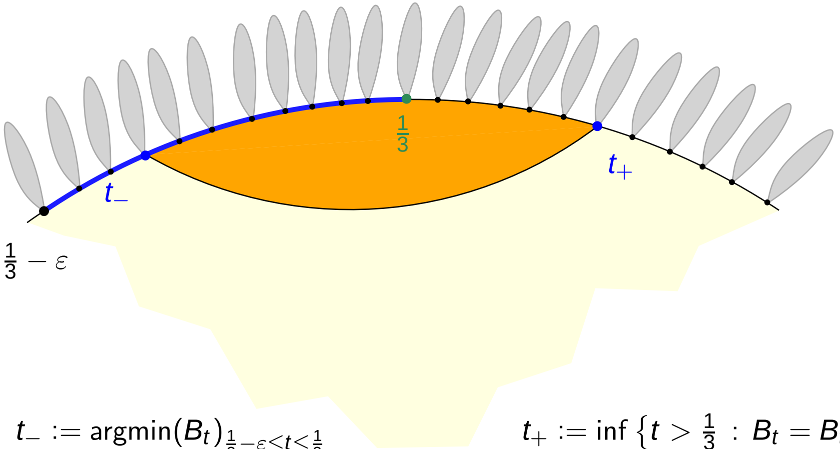
Entrapping the complement of the restriction



Entrapping the complement of the restriction



Entrapping the complement of the restriction



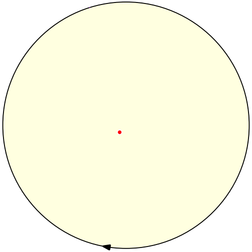
- $t_- := \operatorname{argmin}(B_t)_{\frac{1}{3}-\varepsilon \le t \le \frac{1}{3}}$ $\quad t_+ := \inf\left\{t > \frac{1}{3} \;:\; B_t = B_{t_-}\right\}$
- Asymptotically, $\mathrm{d}_{\mathrm{GH}}\big(a_n Y_n, a_n \mathcal{R}_n^{\varepsilon}(Y_n)\big)$ bounded by range width of (rescaled) labels on $[t_-, t_+]$. *arbitrarily small as $\varepsilon \to 0$*

Resampling argument

Proposition (Leftover is small in model under study)

For $\delta > 0$, $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\Big(d_{GH}\big(a_n X_n, a_n \mathcal{R}_n^{\varepsilon}(X_n)\big) > \delta\Big) = 0.$

- We define a second restriction going backward along the boundary.



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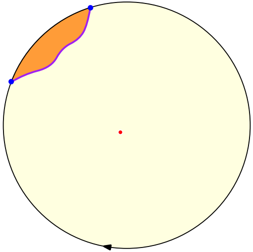
- We define a second restriction going backward along the boundary.
  - Restriction $\mathcal{R}_n^{\varepsilon}$.



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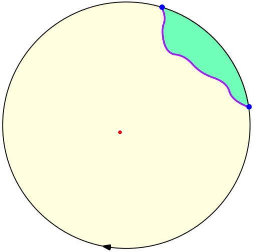
- We define a second restriction going backward along the boundary.
  - Restriction $\mathcal{R}_n^{\varepsilon}$.
  - Restriction $\mathcal{R}_n'^{\varepsilon}$.



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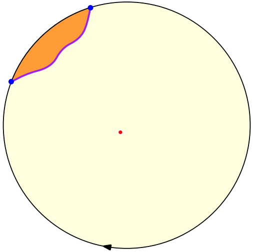
Proposition (Leftover is small in model under study)

For $\delta > 0$, $\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \mathbb{P}\Big(d_{GH}\big(a_n X_n, a_n \mathcal{R}_n^{\varepsilon}(X_n)\big) > \delta \Big) = 0.$

- We define a second restriction going backward along the boundary.
  - Restriction $\mathcal{R}_n^{\varepsilon}$.
  - Restriction $\mathcal{R}_n^{\prime\varepsilon}$.
- Want $X_n \approx \mathcal{R}_n^{\varepsilon}(X_n)$.



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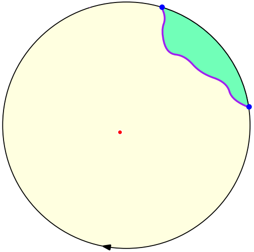
  - Restriction $\mathcal{R}_n'^\varepsilon$.
- Want $X_n \approx \mathcal{R}_n^\varepsilon(X_n)$.
- $\mathcal{R}_n'^\varepsilon(X_n) \approx \mathcal{R}_n'^\varepsilon(Y_n)$



Resampling argument

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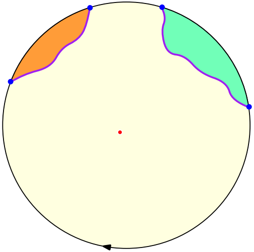

Resampling argument

Proposition (Leftover is small in model under study)

For $\delta > 0$, $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\Big(\mathrm{d_{GH}}\big(a_n X_n, a_n \mathcal{R}_n^\varepsilon(X_n)\big) > \delta\Big) = 0.$

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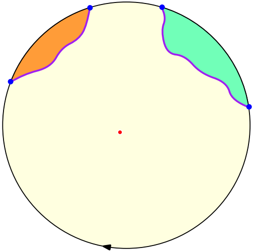
- $\mathcal{R}_n'^\varepsilon(X_n) \approx \mathcal{R}_n'^\varepsilon(Y_n) \approx \mathcal{R}_n^\varepsilon \mathcal{R}_n'^\varepsilon(Y_n) \approx \mathcal{R}_n^\varepsilon \mathcal{R}_n'^\varepsilon(X_n)$

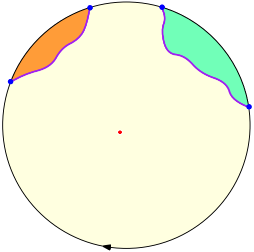


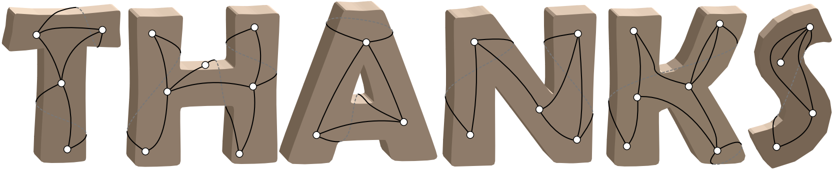
Resampling argument

Proposition (Leftover is small in model under study)

For $\delta > 0$, $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \mathbb{P}\Big(\mathrm{d_{GH}}\big(a_n X_n, a_n \mathcal{R}_n^\varepsilon(X_n)\big) > \delta\Big) = 0.$

- We define a second restriction going backward along the boundary.
  - Restriction $\mathcal{R}_n^\varepsilon$.
  - Restriction $\mathcal{R}_n'^\varepsilon$.
- Want $X_n \approx \mathcal{R}_n^\varepsilon(X_n)$.
- $\mathcal{R}_n'^\varepsilon(X_n) \approx \mathcal{R}_n'^\varepsilon(Y_n) \approx \mathcal{R}_n^\varepsilon \mathcal{R}_n'^\varepsilon(Y_n) \approx \mathcal{R}_n^\varepsilon \mathcal{R}_n'^\varepsilon(X_n)$
- Thus $X_n \approx \mathcal{R}_n^\varepsilon(X_n)$.




THANKS