Scaling limit and bijective combinatorics of maps

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presentation for “habilitation à diriger des recherches”

June 7, 2022

What is a map?

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Village de la Putain
Temple du Rock
Citadelle des impôts
Grotte satanique
La cité de la Mort
La cité perdue
Taste of ink Tattoo
Île du naufragé
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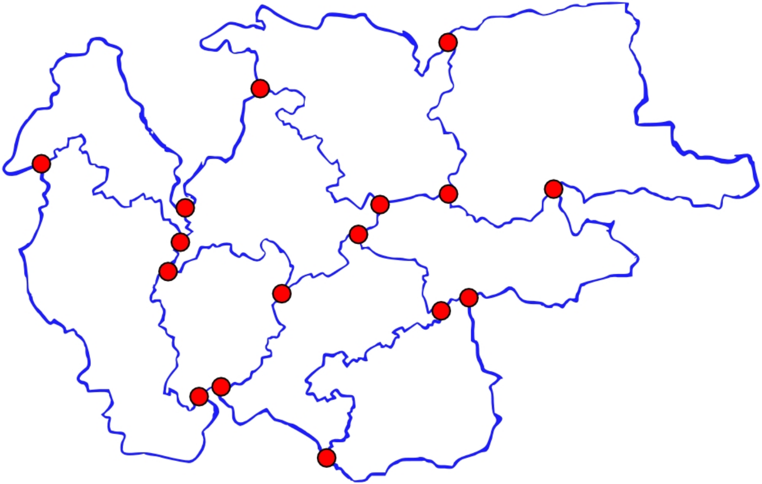

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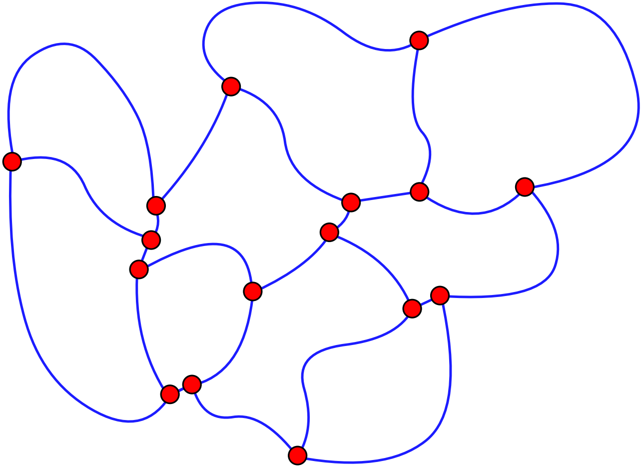

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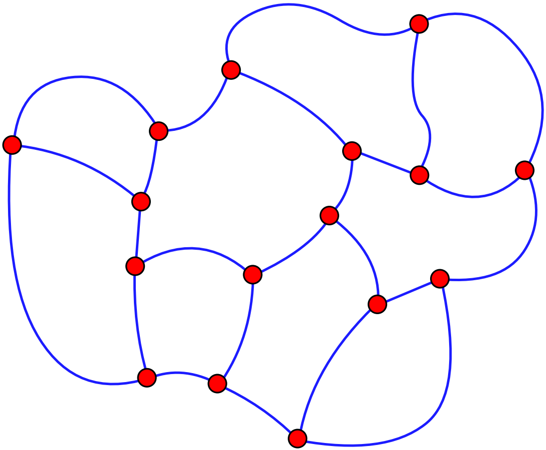
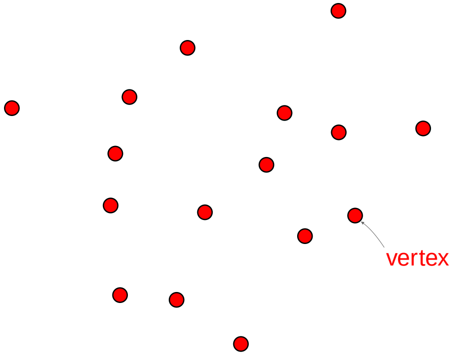


Plane map, formally



- We have **vertices**

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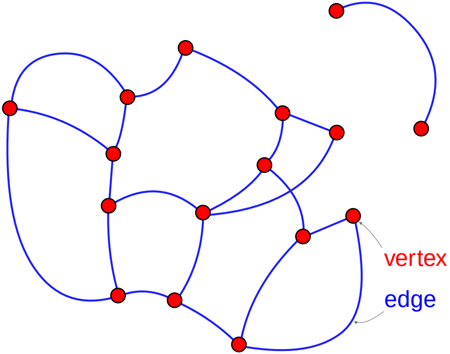

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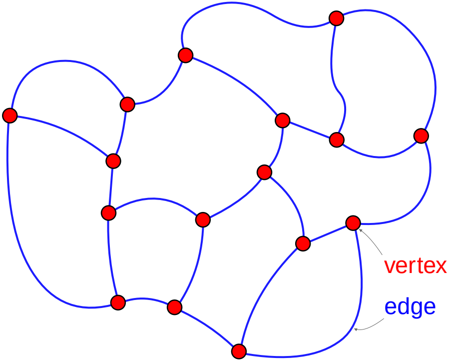

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  - in a connected way.

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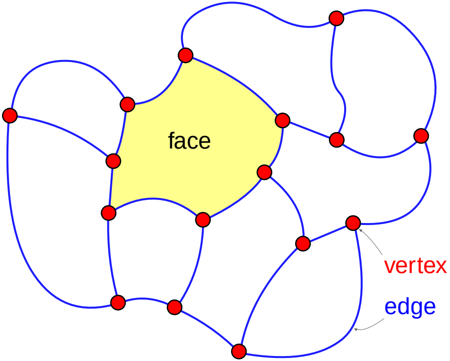

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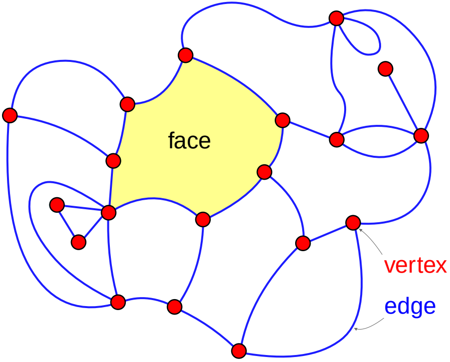

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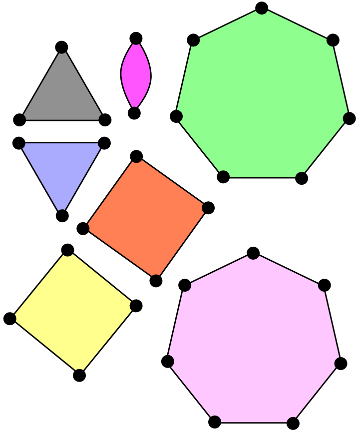
- Multiple edges and loops are allowed.

Other point of view: gluing of polygons



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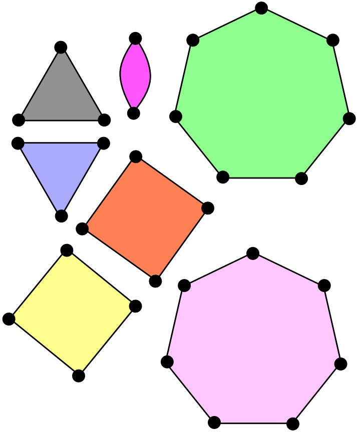

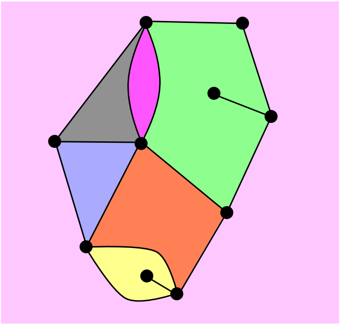

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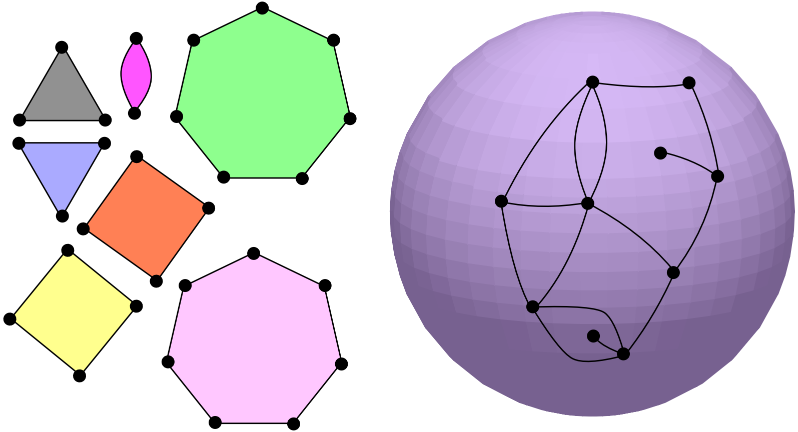
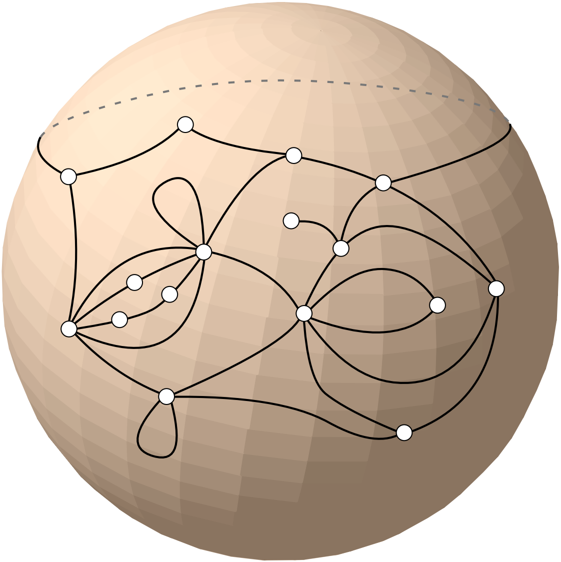


Root = distinguished corner



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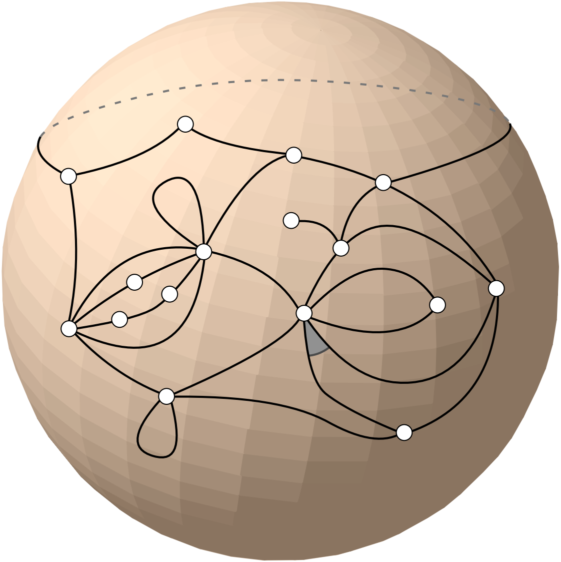
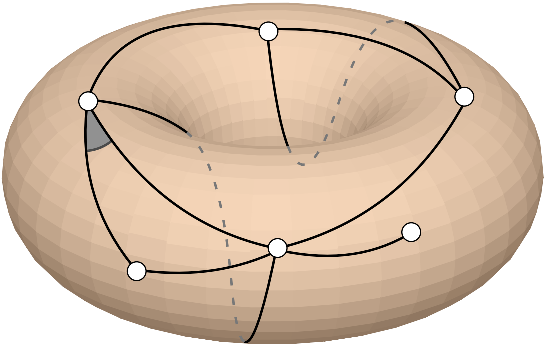


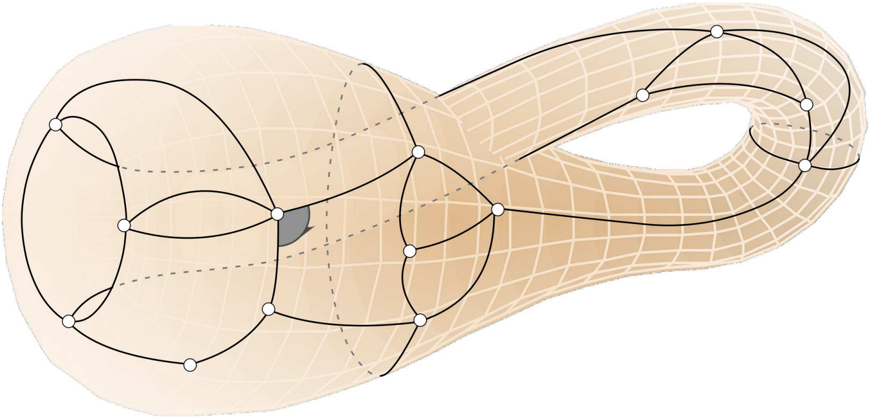
Genus g maps



genus g map: graph embedded in the surface of genus g, in such a way that the faces are homeomorphic to disks

maps are defined up to direct homeomorphism of the underlying surface

Nonorientable maps



root: distinguished corner given with a local orientation

maps are defined up to homeomorphism of the underlying surface

Why study maps?

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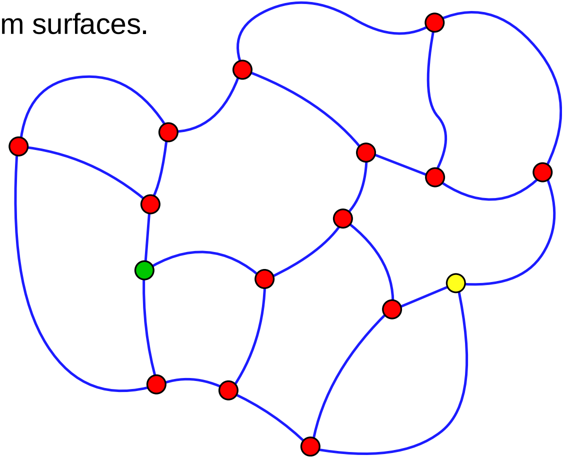

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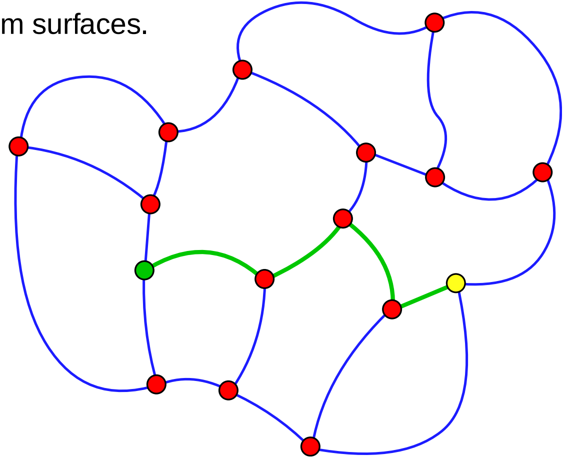

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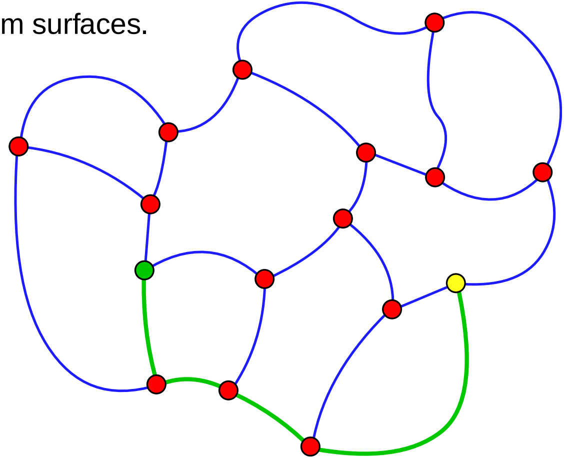

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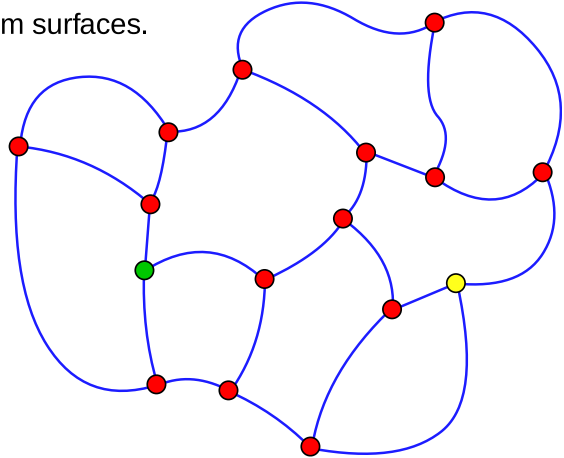

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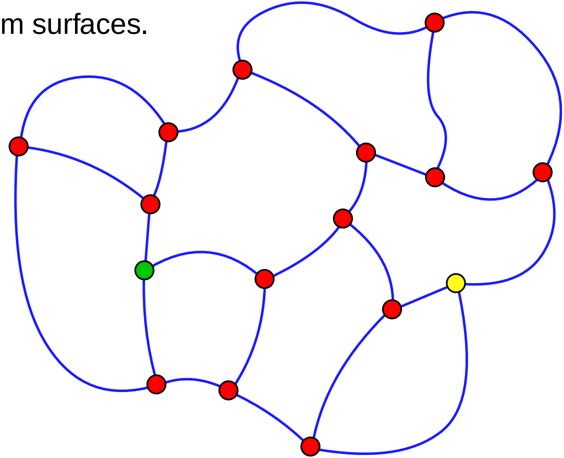
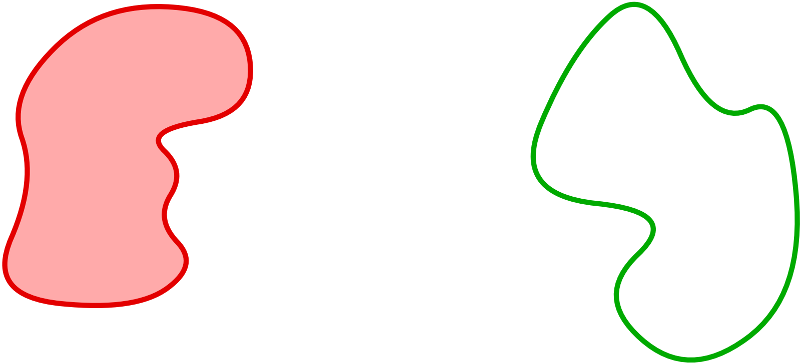


- They are beautiful!
- They share deep links with trees.

Gromov–Hausdorff topology



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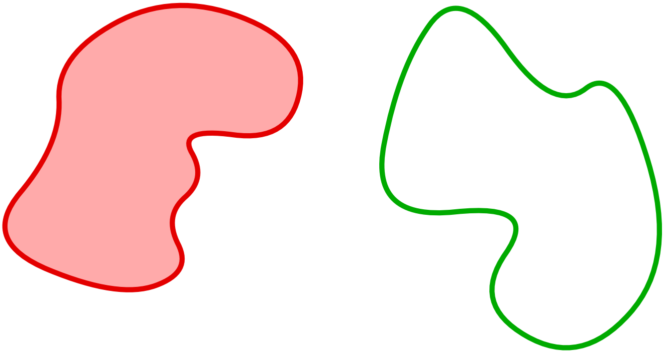

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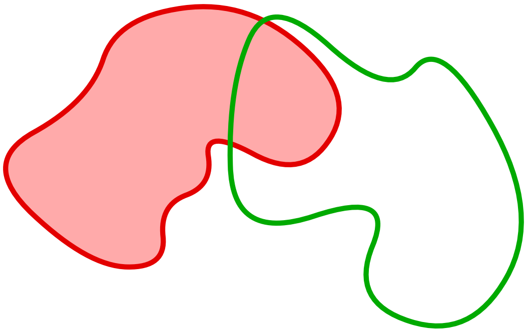

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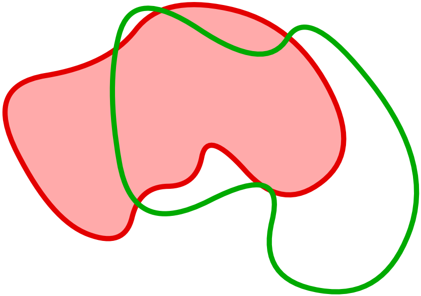

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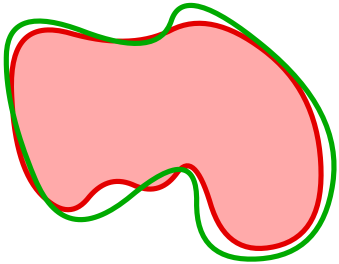

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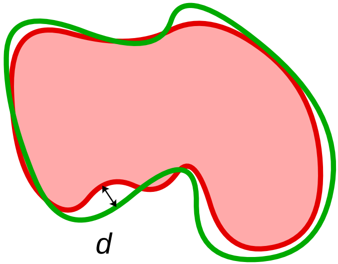

The Brownian sphere

- $a\,\mathbf{m}$: finite metric space obtained by endowing the vertex-set of $\mathbf{m}$ with a times the graph metric (each edge has length a).

Theorem (Le Gall '11, Miermont '11)

Let $\mathbf{q}_n$ be a uniform plane quadrangulation with n faces. The sequence $\left((8n/9)^{-1/4}\mathbf{q}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space called the Brownian sphere.

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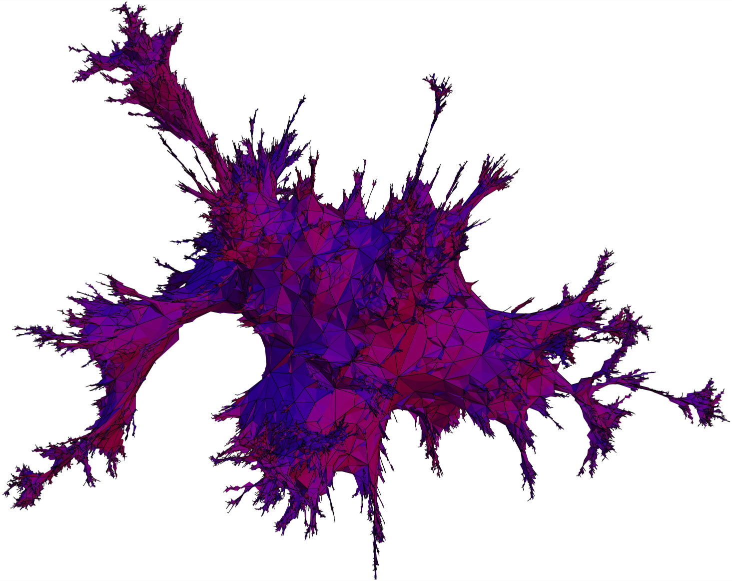
Definition (Convergence for the Gromov–Hausdorff topology)

A sequence $(\mathcal{X}_n)$ of compact metric spaces **converges in the sense of the Gromov–Hausdorff topology** toward a metric space $\mathcal{X}$ if there exist isometric embeddings $\varphi_n : \mathcal{X}_n \to \mathcal{Z}$ and $\varphi : \mathcal{X} \to \mathcal{Z}$ into a common metric space $\mathcal{Z}$ such that $\varphi_n(\mathcal{X}_n)$ converges toward $\varphi(\mathcal{X})$ in the sense of the Hausdorff topology.

Uniform plane quadrangulation with 50 000 faces



Earlier results

- [Chassaing–Schaeffer '04]
  - the scaling factor is $n^{1/4}$
  - scaling limit of functionals of random uniform quadrangulations (radius, profile)
- [Marckert–Mokkadem '06]
  - introduction of the Brownian sphere (called Brownian map)
- [Le Gall '07]
  - the sequence of rescaled quadrangulations is relatively compact
  - any subsequential limit has the topology of the Brownian sphere
  - any subsequential limit has Hausdorff dimension 4
- [Le Gall–Paulin '08], [Miermont '08]
  - the topology of any subsequential limit is that of the two-sphere
- [Bouttier–Guitter '08]
  - limiting joint distribution between three uniformly chosen vertices

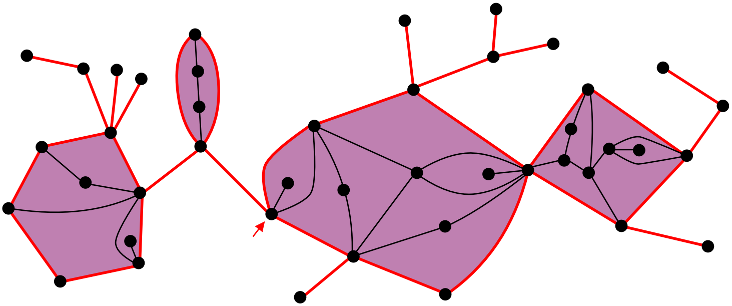
Universality of the Brownian sphere

Many other natural models of plane maps converge to the Brownian sphere (up to a model-dependent scale constant):

$$c\, n^{-1/4}\, \mathbf{m}_n \xrightarrow[n\to\infty]{} \text{Brownian sphere.}$$

- [Le Gall '11] uniform p-angulations for $p \in \{3, 4, 6, 8, 10, \dots\}$ and Boltzmann bipartite maps with fixed number of vertices
- [Beltran and Le Gall '12] quadrangulations with no pendant edges
- [Addario-Berry–Albenque '13] simple triangulations and simple quadrangulations
- [B.–Jacob–Miermont '14] maps with fixed number of edges
- [Abraham '14] bipartite maps with fixed number of edges
- [Marzouk '17] bipartite maps with prescribed degree sequence
- [Curien–Le Gall '19] random length plane triangulations
- [Addario-Berry–Albenque '20] p-angulations for odd $p \geq 5$
- [Marzouk '20] planes bipartites maps with prescribed degrees

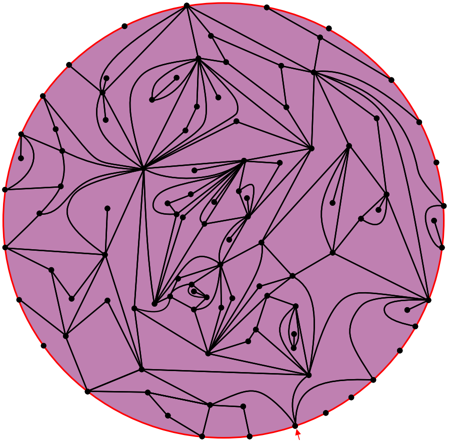
Plane quadrangulation with a boundary



plane map whose faces have degree 4, except possibly the root face

the boundary is not necessarily a simple curve

Plane quadrangulation with a simple boundary



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Brownian disks

- $\mathbf{q}_{n,p}$ uniform among quadrangulations with a boundary having area n and perimeter p
- $\ell_n/\sqrt{2n} \to L \in (0, \infty)$

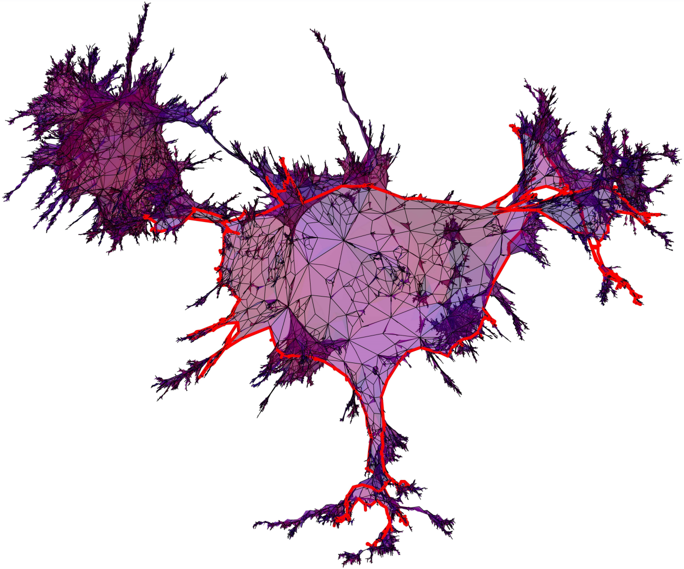
Theorem (B.–Miermont '15)

The sequence $\left((8n/9)^{-1/4}\mathbf{q}_{n,2\ell_n}\right)_{n\geq 1}$ *converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space* $\mathbf{BD}_L$ *called the Brownian disk of perimeter* L.

Theorem (B. '11)

Let $L > 0$ *be fixed. Almost surely, the space* $\mathbf{BD}_L$ *is homeomorphic to the closed unit disk of* $\mathbb{R}^2$. *Moreover, almost surely, the Hausdorff dimension of* $\mathbf{BD}_L$ *is* 4, *while that of its boundary* $\partial\mathbf{BD}_L$ *is* 2.

40 000 faces and boundary length 1 000



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- [B.–Miermont '15] $2p$-ang., uniform bip. maps, bip. Boltzmann maps
- [Gwynne–Miller '19] Boltzmann quad. with a simple boundary
- [Albenque–Holden–Sun '20] Boltzmann tri. with a simple boundary

Degenerate regimes

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Theorem (B. '11) $\ell_n/\sqrt{2n} \to 0$

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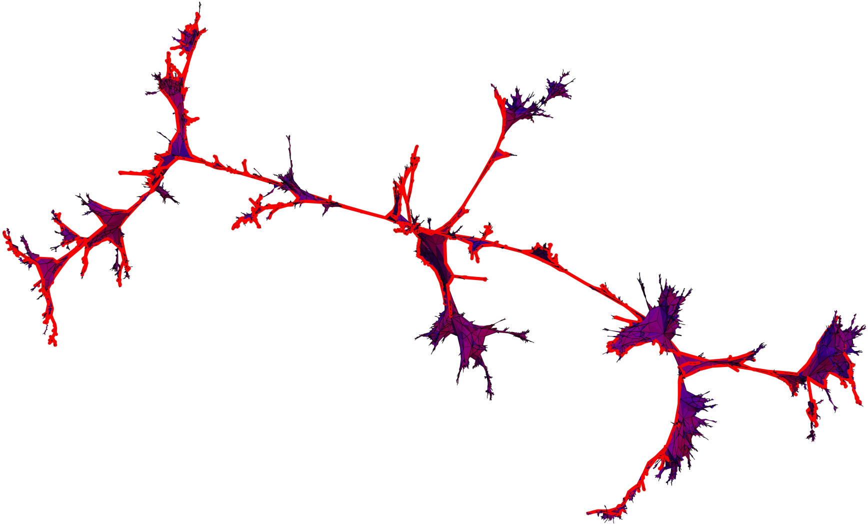
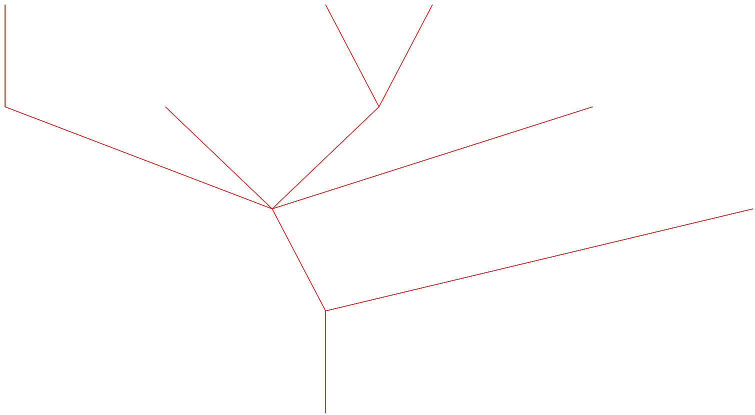
- [Bouttier–Guitter '09] computation of the two-point function
- [Marzouk '20] bipartite maps with prescribed degrees

10 000 faces and boundary length 2 000



The Continuum Random Tree



tree

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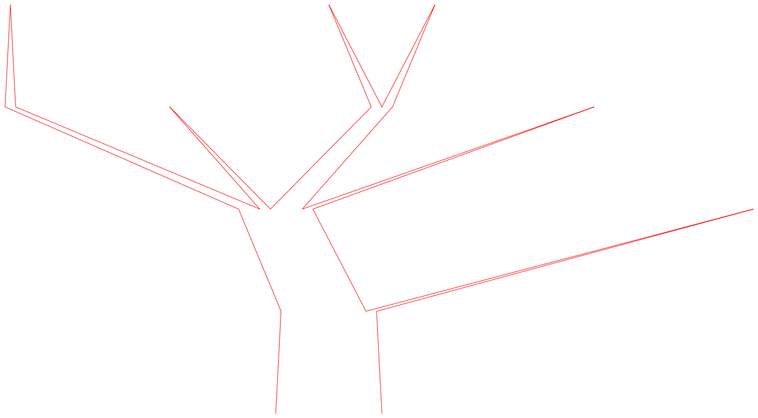

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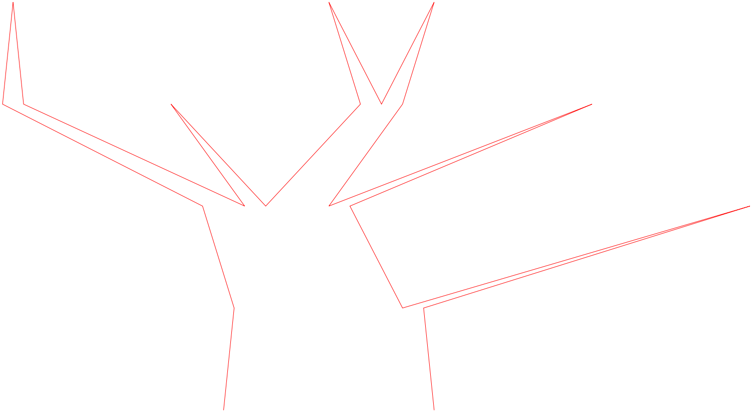

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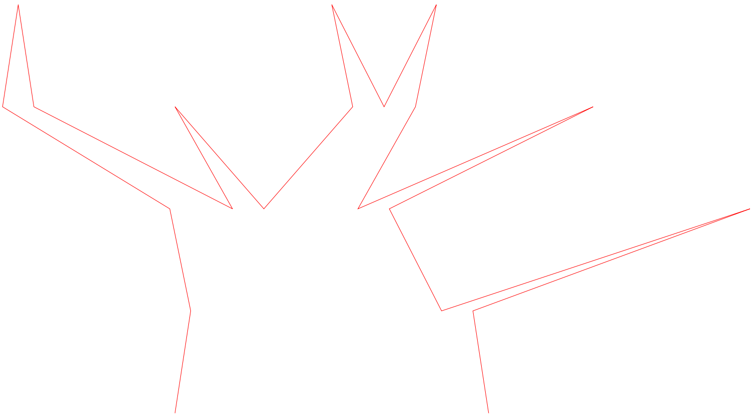

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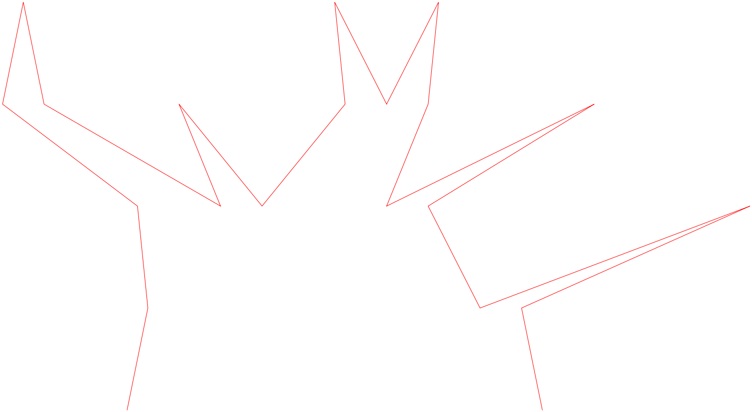

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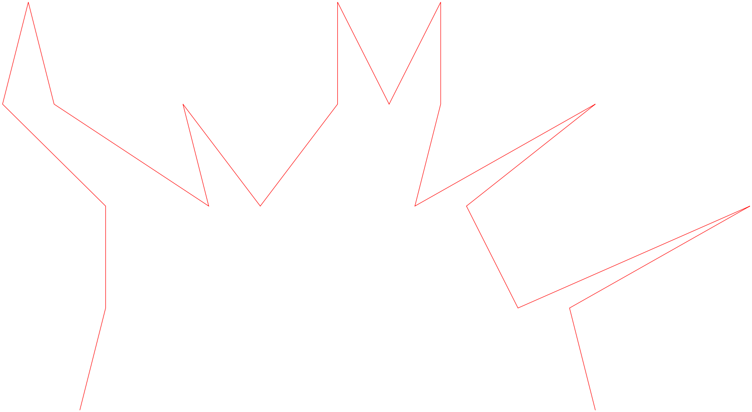

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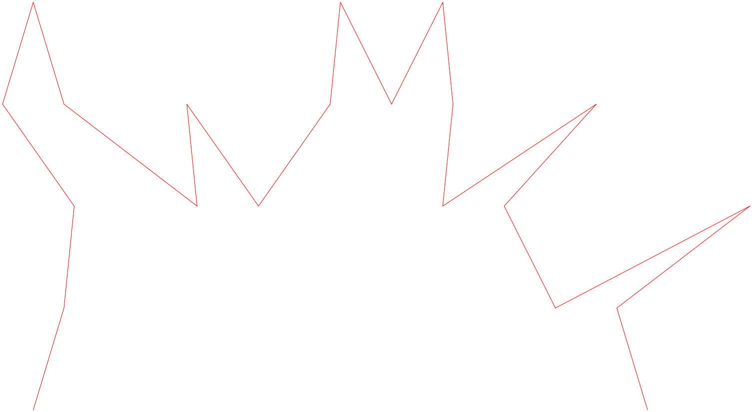

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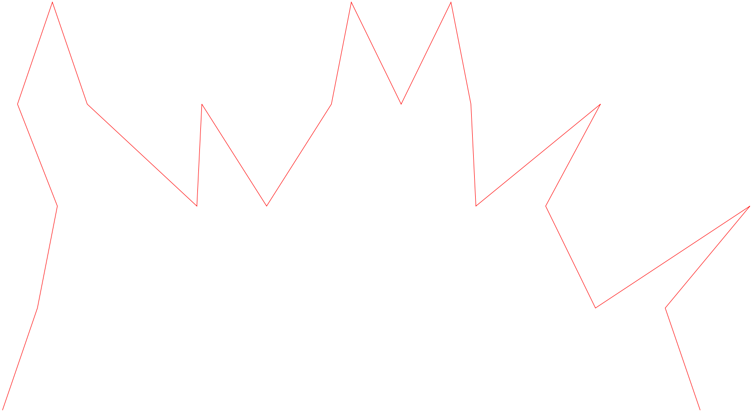

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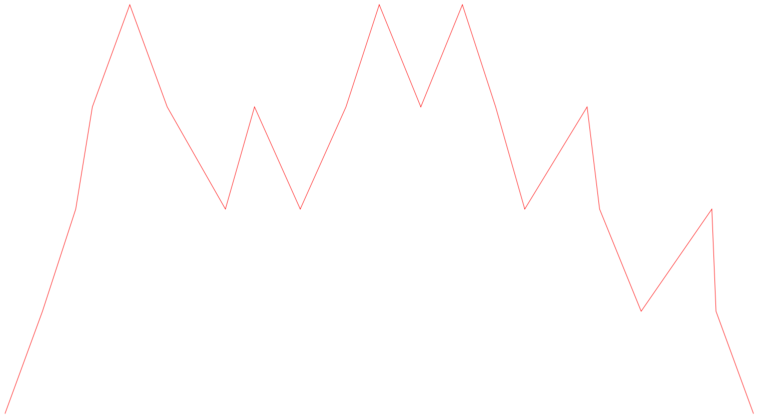

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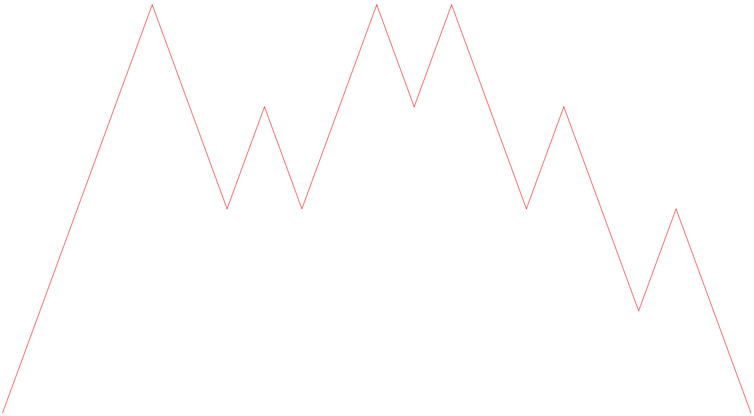
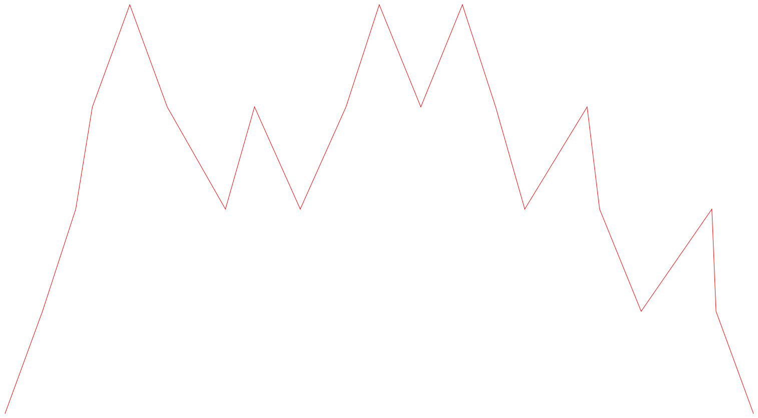


Dyck path

The Continuum Random Tree



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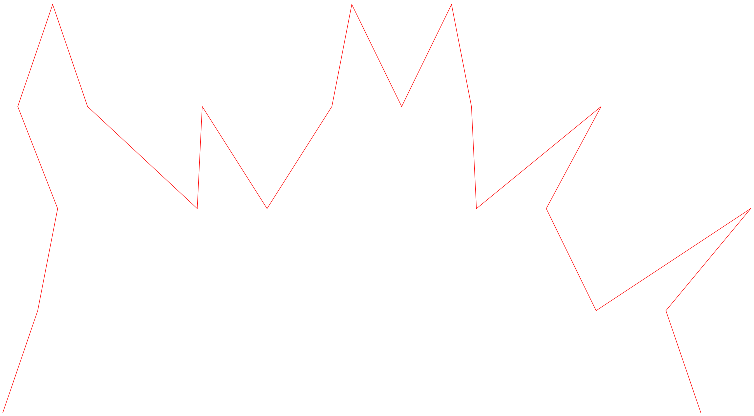

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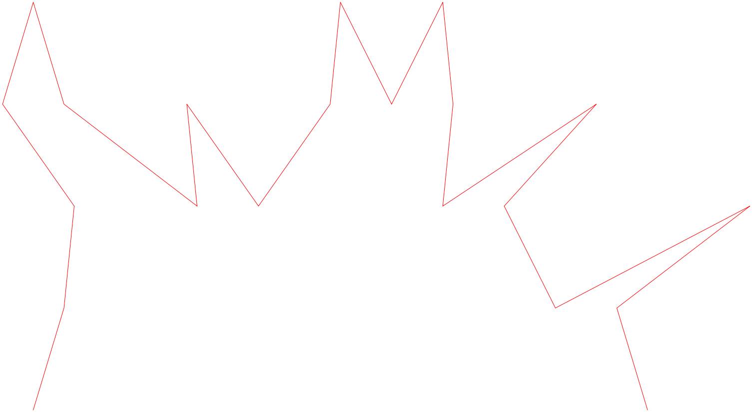

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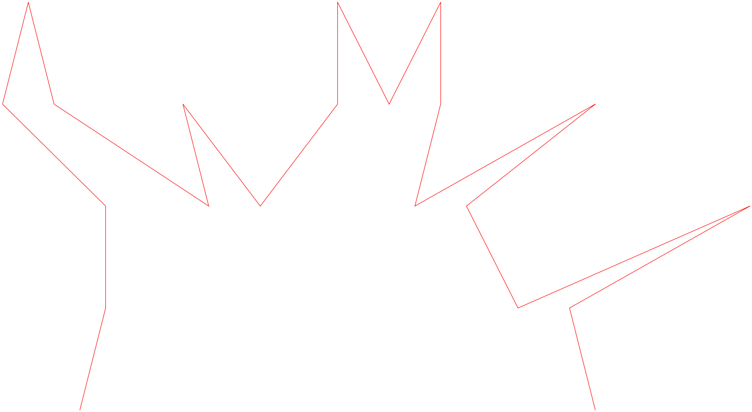

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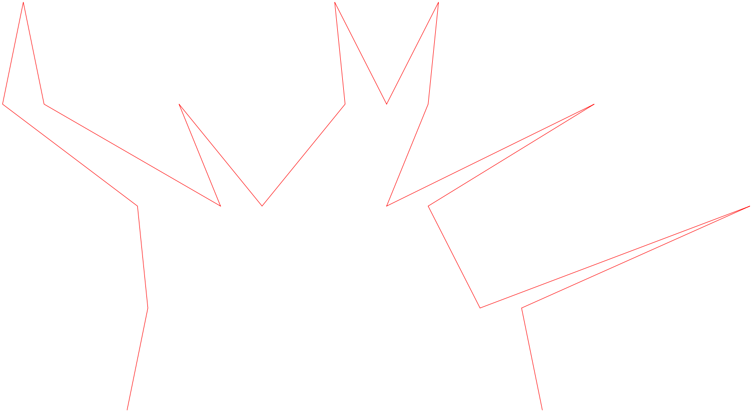

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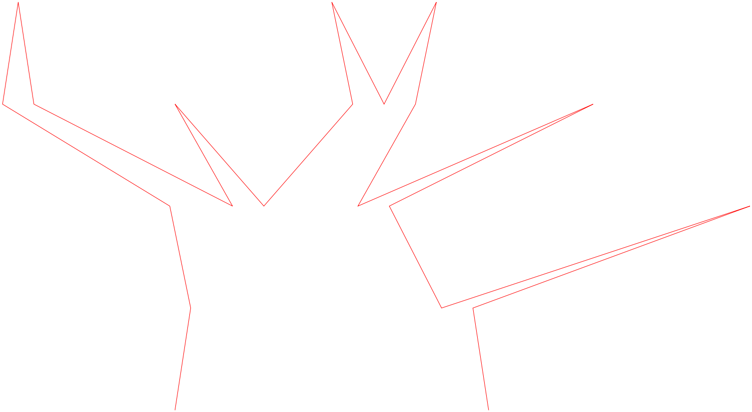
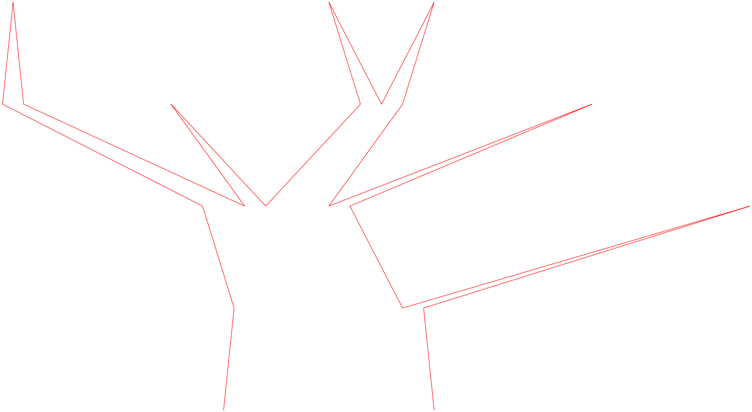
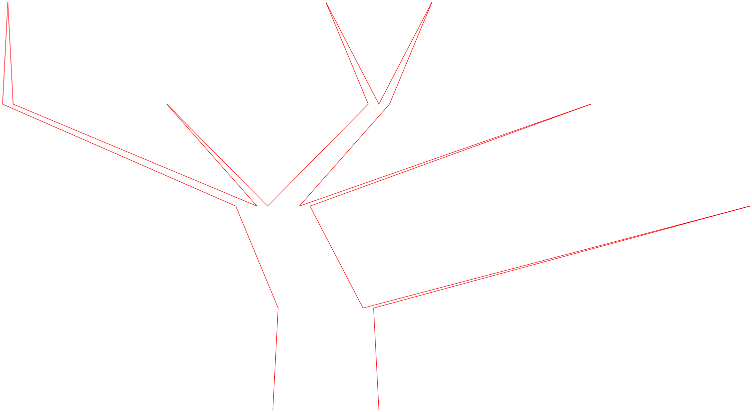


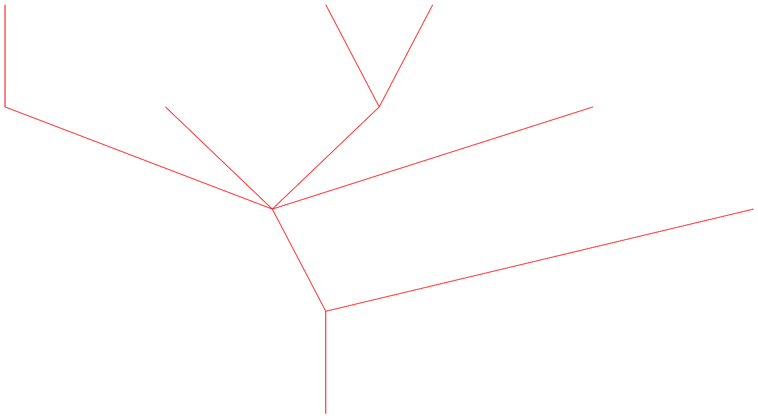
The Continuum Random Tree



The Continuum Random Tree

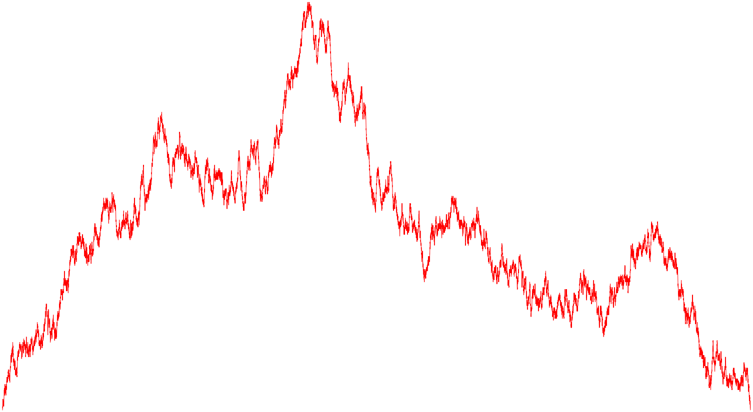


The Continuum Random Tree



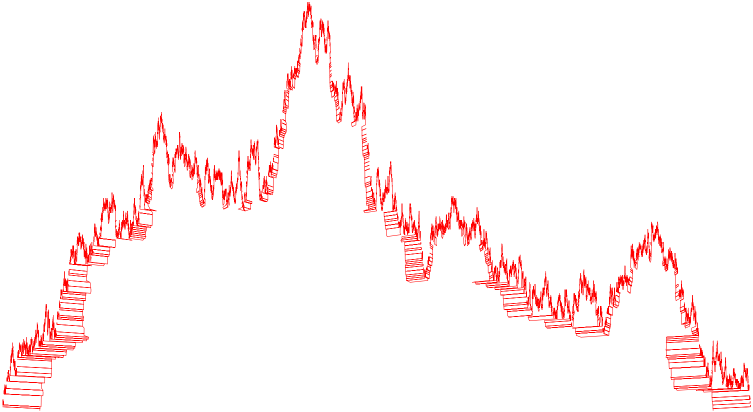
tree

The Continuum Random Tree

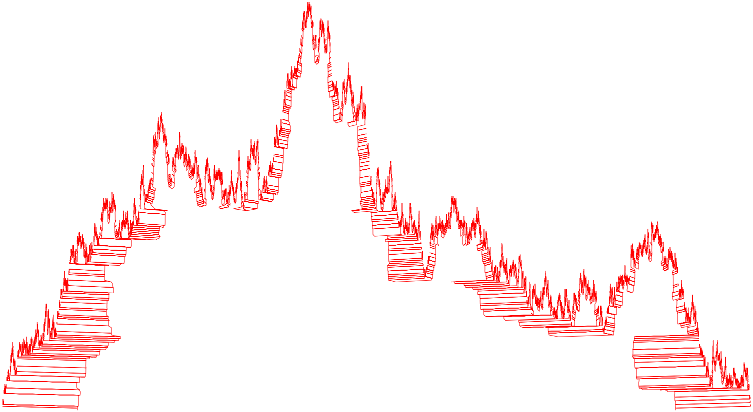


Brownian excursion: Brownian motion on $[0, 1]$ conditioned to stay positive on $(0, 1)$ and be back at 0 at time 1

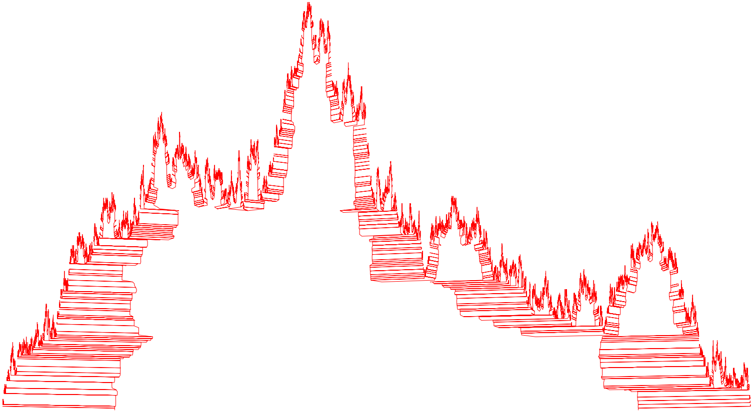
The Continuum Random Tree



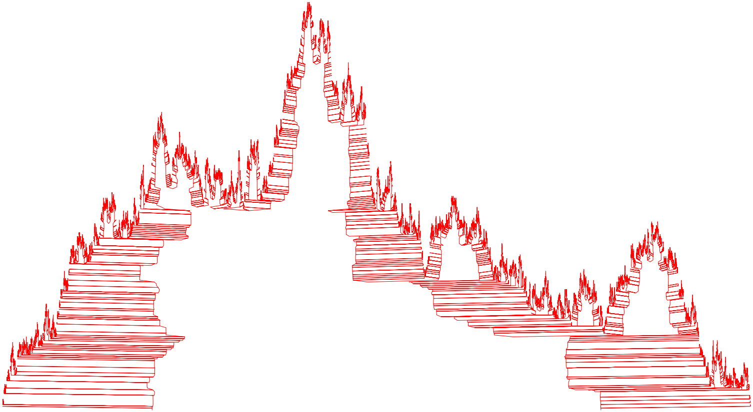
The Continuum Random Tree



The Continuum Random Tree



The Continuum Random Tree



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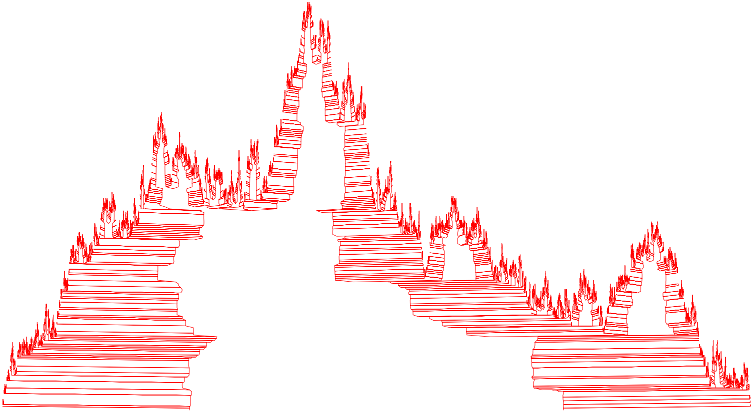
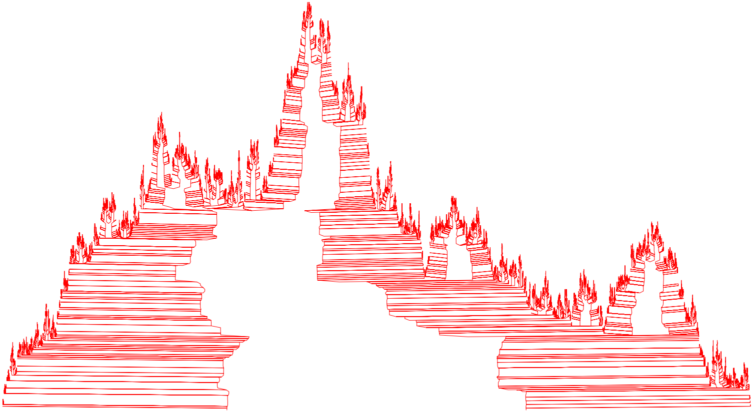
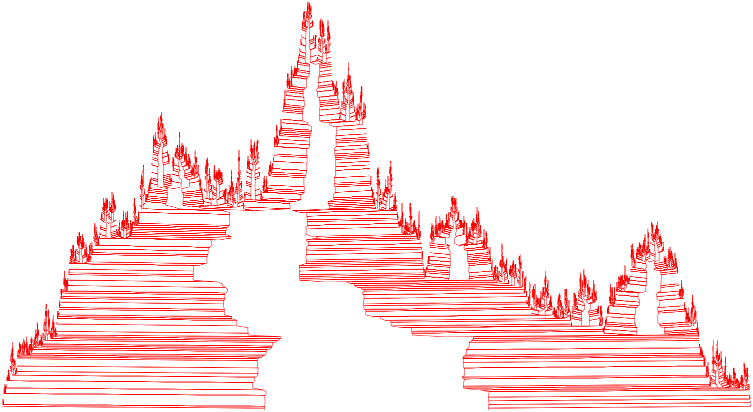


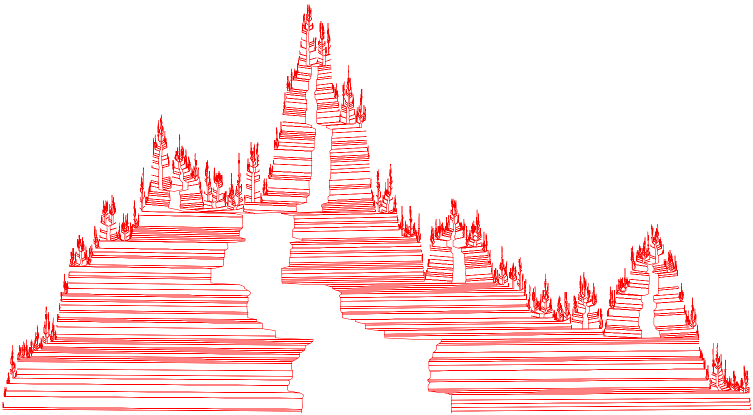
The Continuum Random Tree



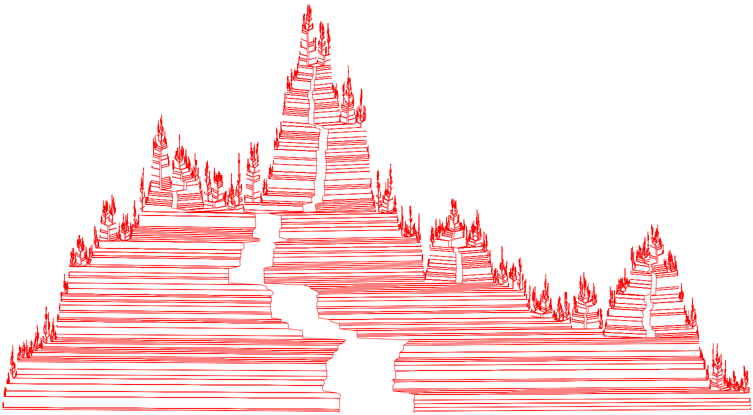
The Continuum Random Tree



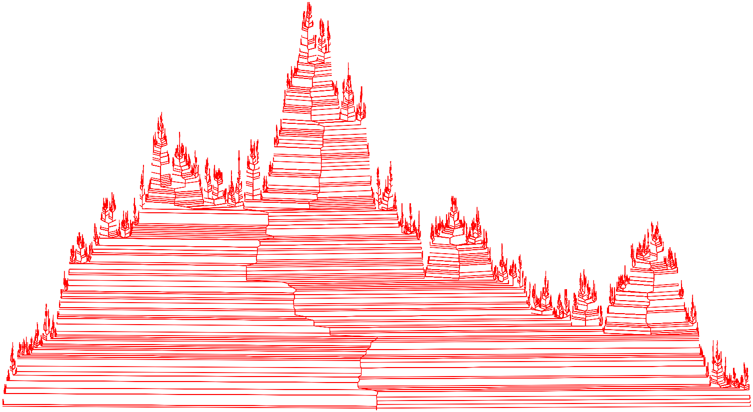
The Continuum Random Tree



The Continuum Random Tree

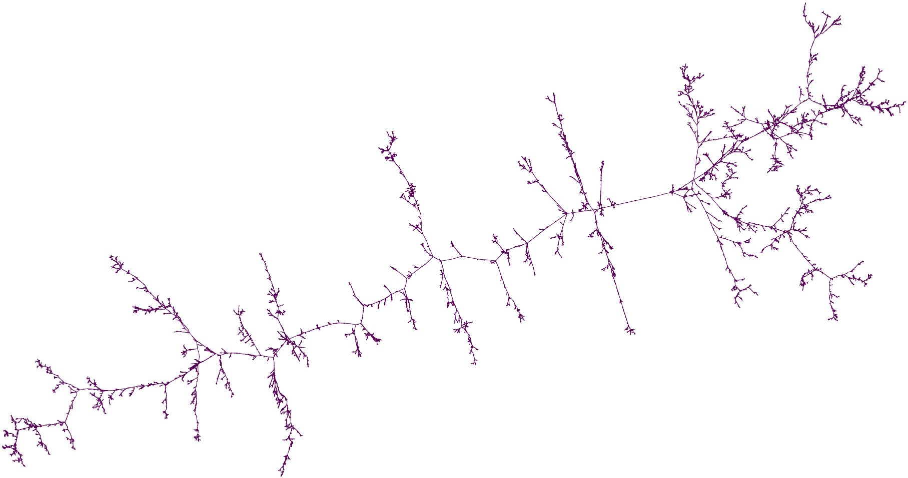


The Continuum Random Tree



CRT

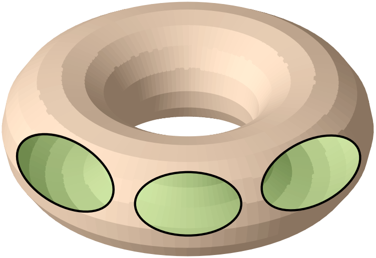
30 000 edges



Surface with a boundary

Definition

Let $\Sigma^{\partial}_{g,p}$ denote the surface with a boundary constructed by removing p open disks from the connected sum of g tori.



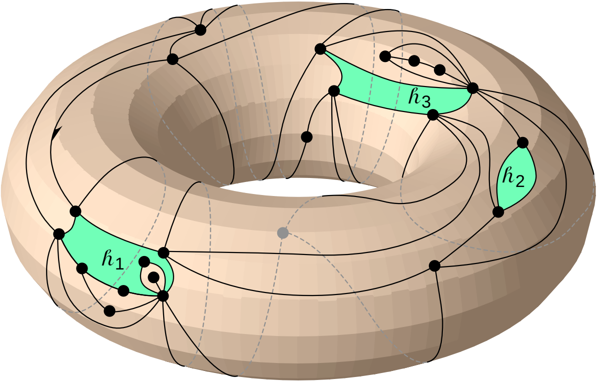
Classification theorem

Every compact, connected and orientable surface with a boundary is homeomorphic to a unique $\Sigma^{\partial}_{g,p}$.

Quadrangulation with holes

Definition

A quadrangulation with p holes is a rooted bipartite map with p distinguished faces $h_1, \ldots, h_p$ and whose other faces are of degree 4.



Brownian surfaces

- $g \geq 0$, $p \geq 0$ fixed integers
- $\ell_n^i/\sqrt{2n} \to L^i \in (0,\infty)$ for $1 \leq i \leq p$
- $\mathbf{q}_n$ uniform among $\mathbf{Q}^{[g]}_{n,(\ell_n^1,\dots,\ell_n^p)}$

Brownian surfaces

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genus

perimeters of the p holes

number of quadrangles

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genus

perimeters of the p holes

number of quadrangles

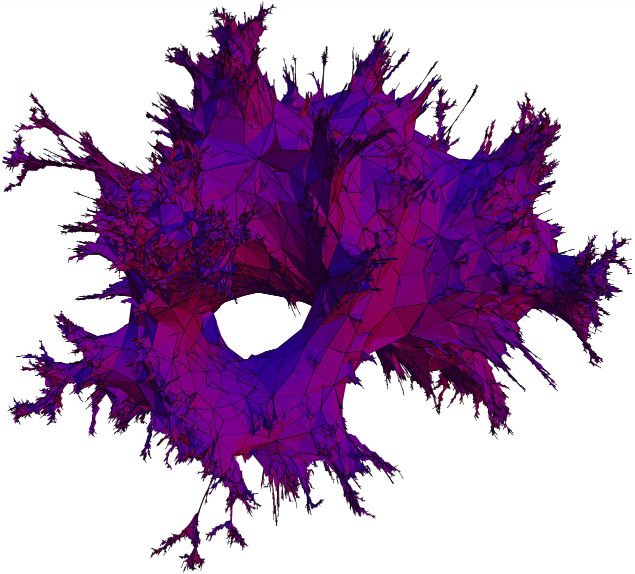
Theorem (B.–Miermont '15 & '22) $(g,p) \neq (0,0)$

The sequence $\left((8n/9)^{-1/4}\mathbf{q}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space called the Brownian surface of genus g and perimeter $(L^1,\dots,L^p)$.

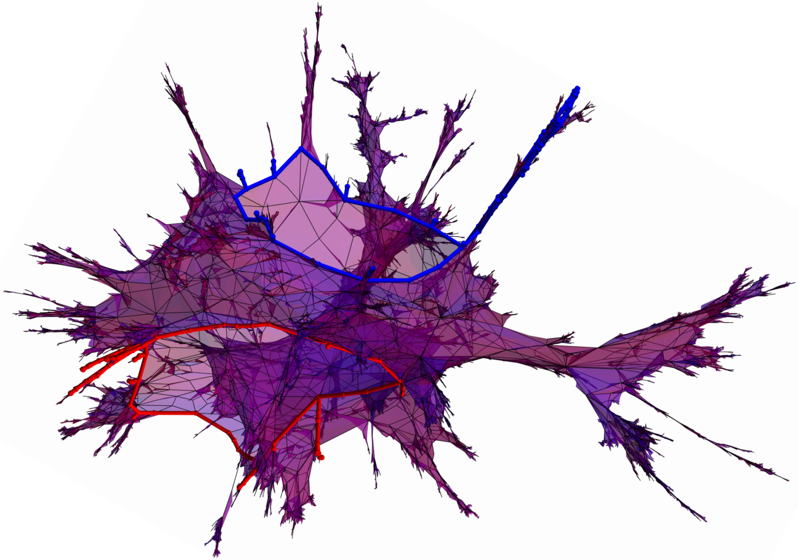
Theorem (B. '16)

Almost surely, the above is homeomorphic to $\Sigma^{\partial}_{g,p}$; its Hausdorff dimension is 4 and that of each of its boundary components is 2.

50 000 faces, genus 1



10 000 faces, genus 1, boundary lengths 60 and 80



Toward Brownian nonorientable surfaces

- $\mathbf{q}_n$ uniform among quadrangulations with n faces of a fixed nonorientable surface

Theorem (Chapuy & Dołęga '17)

Up to extraction, the sequence $\left((8n/9)^{-1/4}\mathbf{q}_n\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space.

A history of bijections

Encoding of pointed maps

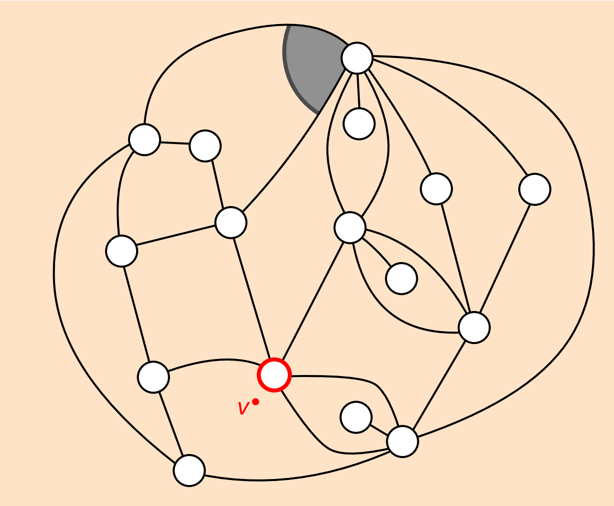
	sphere	orientable	nonorientable
bip. quad.	CVS	CMS	CD
general maps	BDG	BDG + CMS	B

CVS: [Cori–Vauquelin ’81] and [Schaeffer ’98]
BDG: [Bouttier–Di Francesco–Guitter ’04]
CMS: [Chapuy–Marcus–Schaeffer ’09]
CD: [Chapuy–Dołęga ’17]
B: [B. ’22]

Similar bijections

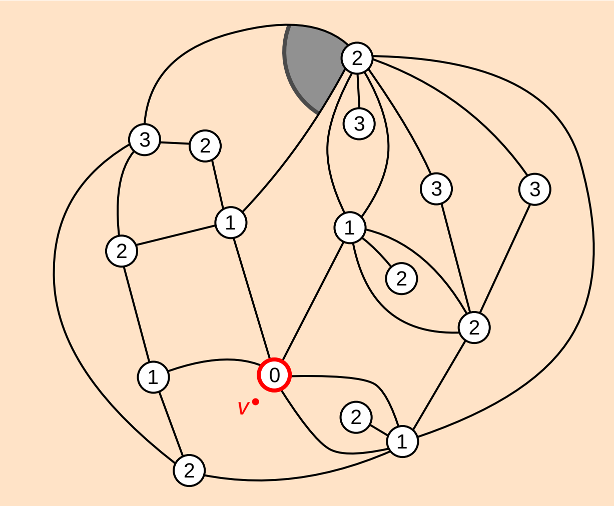
- [Miermont ’09] multi-pointed quadranguations
- [Ambjørn–Budd ’13] CVS with rules inverted

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



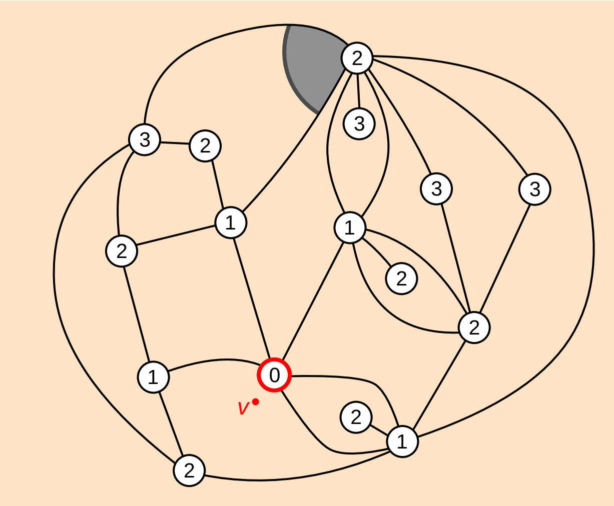
- Start with a pointed bipartite quadrangulation.

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer

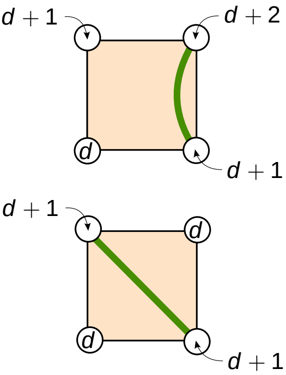


- Start with a pointed bipartite quadrangulation.
- Label the vertices with their distance to $v^\bullet$.

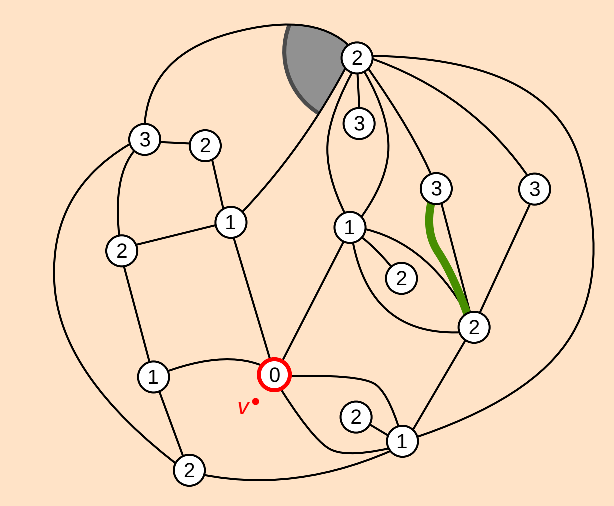
Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



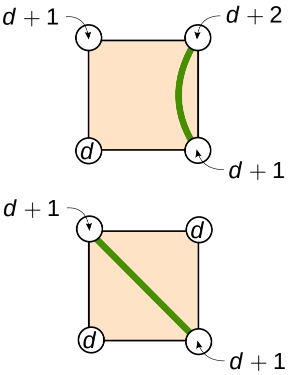
- Apply the rule:



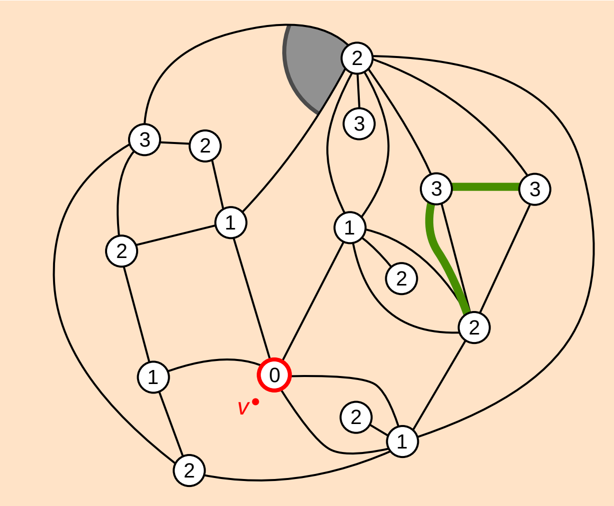
Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



- Apply the rule:



Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



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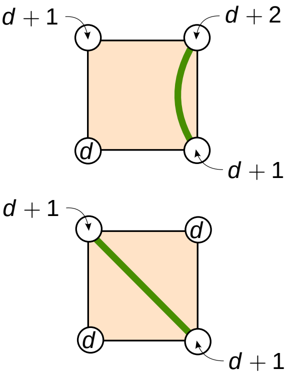
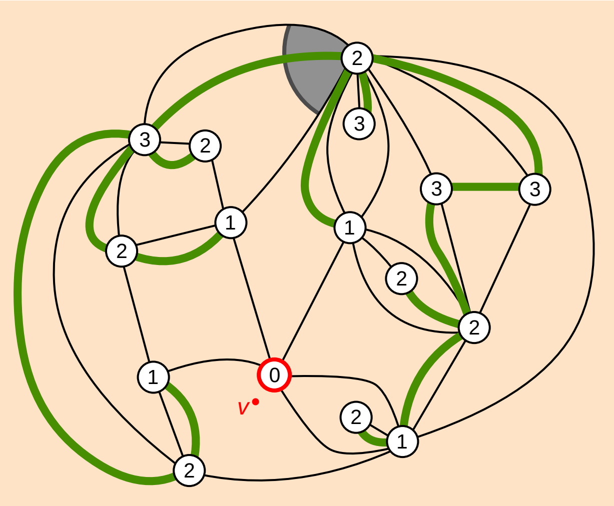


Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



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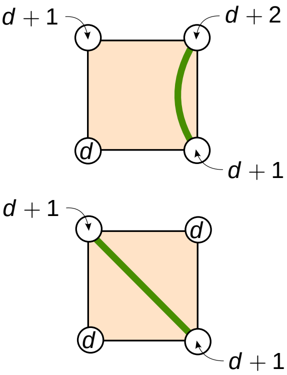
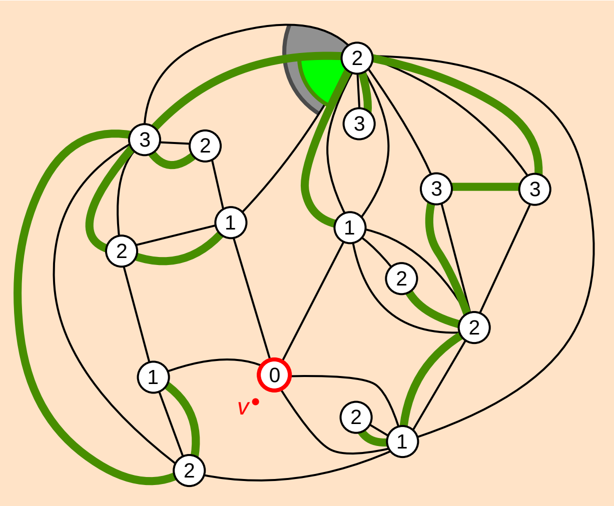
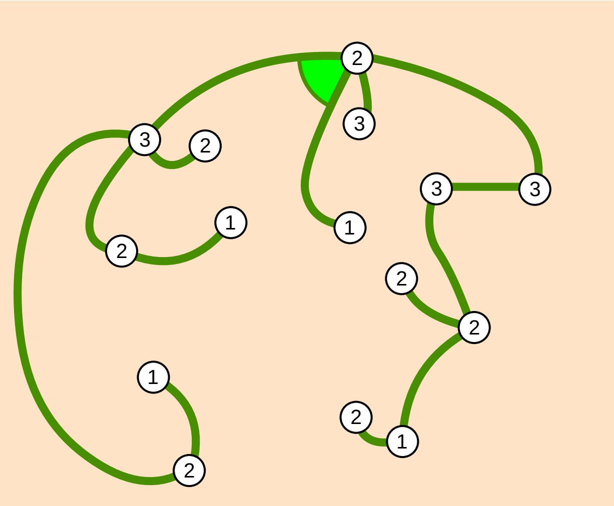


Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



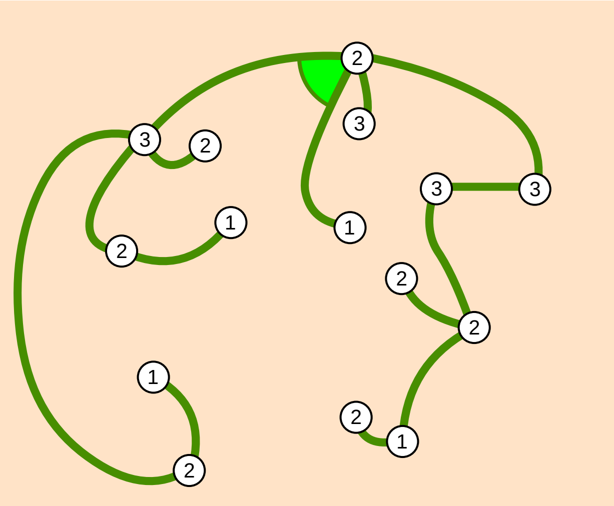
- Start with a pointed bipartite quadrangulation.
- Label the vertices with their distance to $v^\bullet$.
- Apply the rule.
- Root.

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



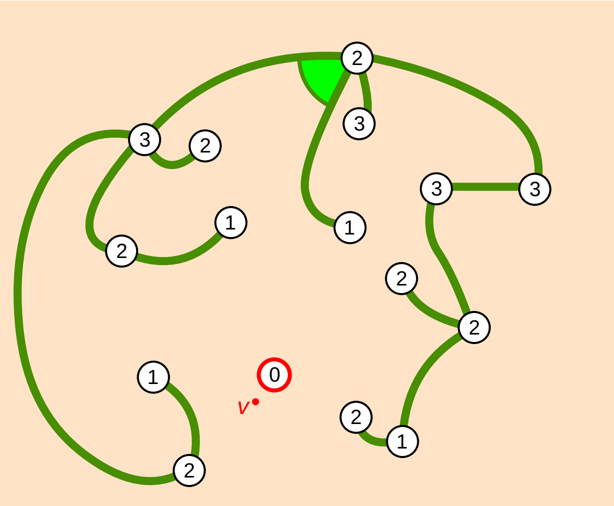
- Start with a pointed bipartite quadrangulation.
- Label the vertices with their distance to $v^\bullet$.
- Apply the rule.
- Root.
- Remove the initial edges and $v^\bullet$.

Inverse construction



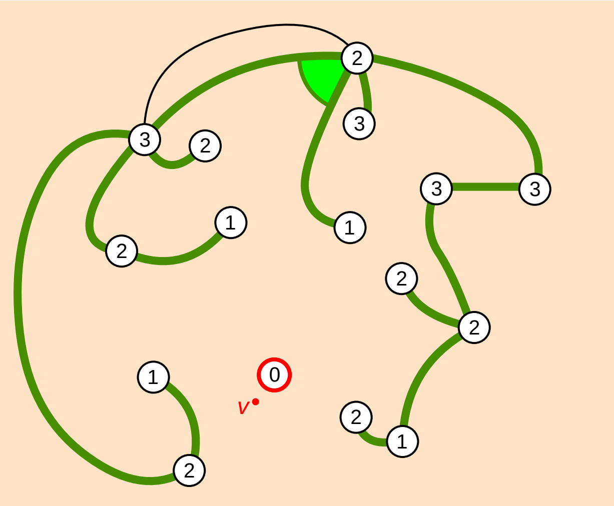
- Take a well-labeled unicellular map.

Inverse construction



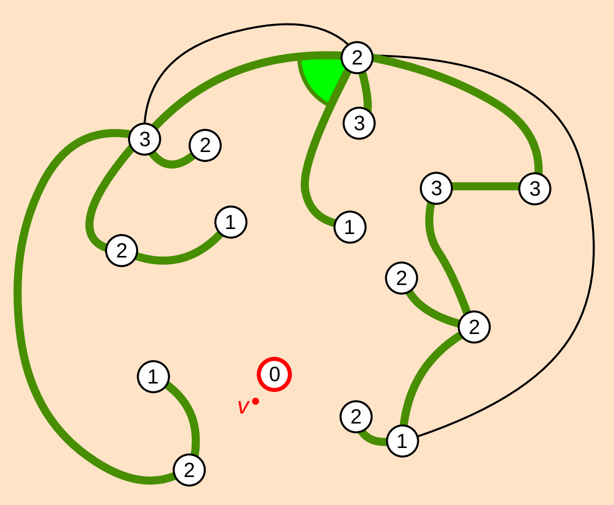
- Take a well-labeled unicellular map.
- Add a vertex $v^\bullet$ inside the unique face.

Inverse construction



- Take a well-labeled unicellular map.
- Add a vertex $v^\bullet$ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.

Inverse construction



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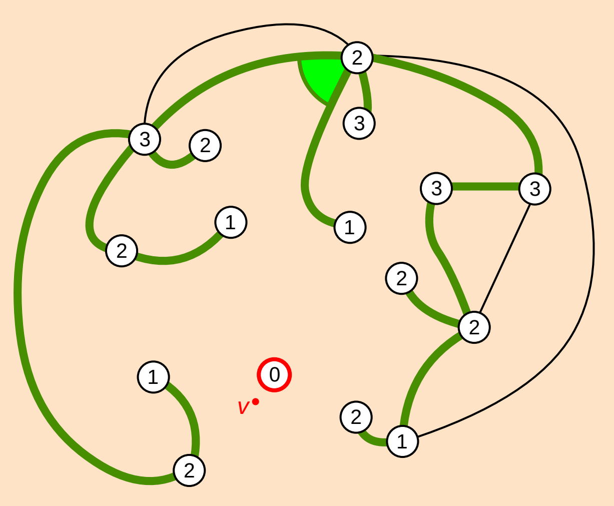

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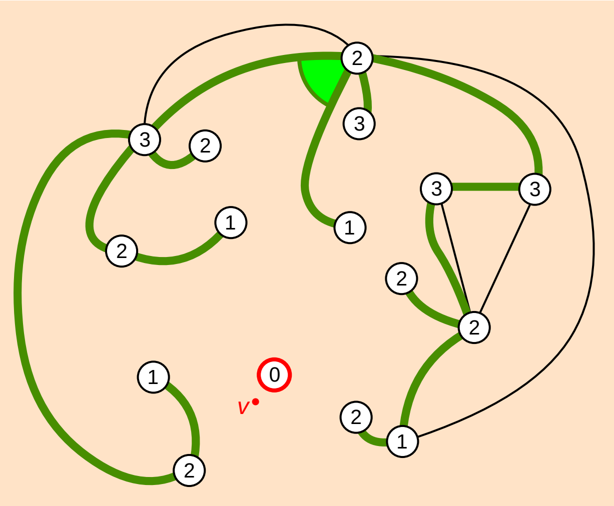

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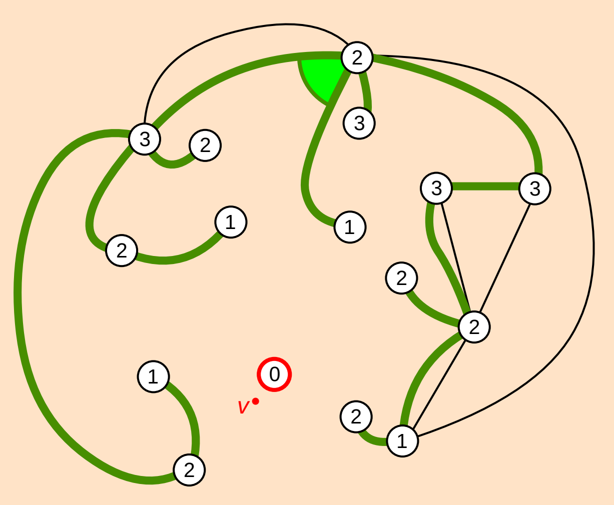

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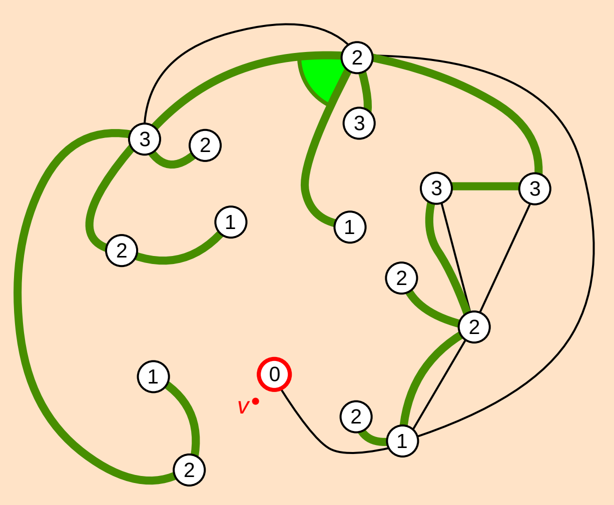

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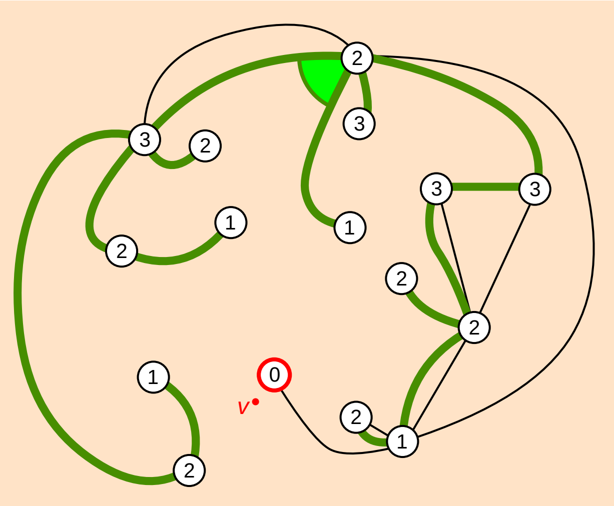

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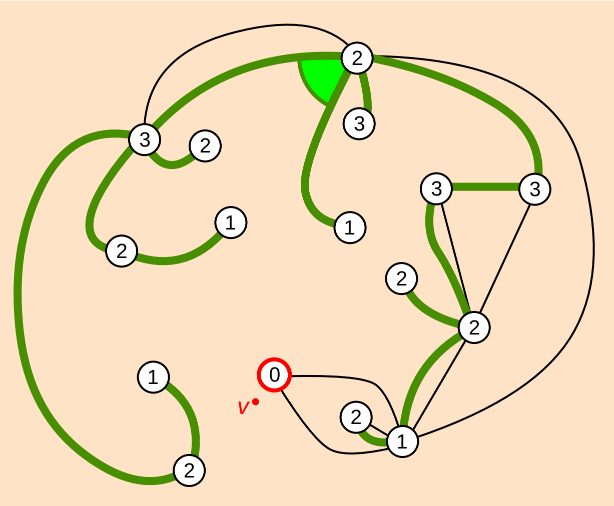

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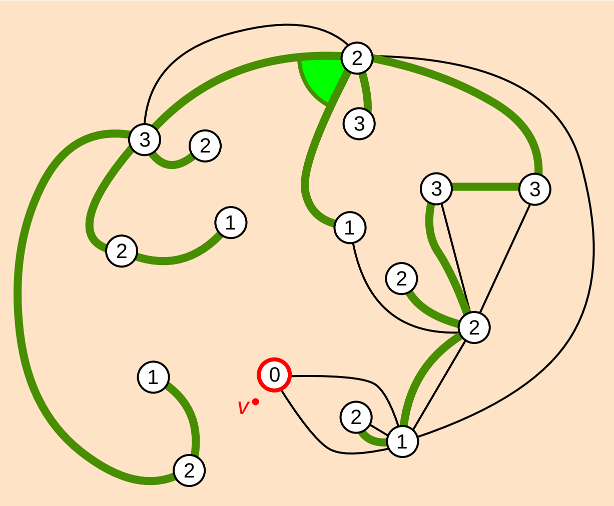

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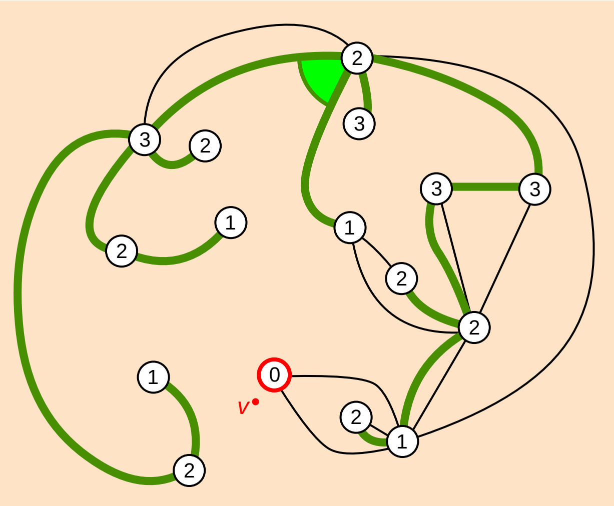

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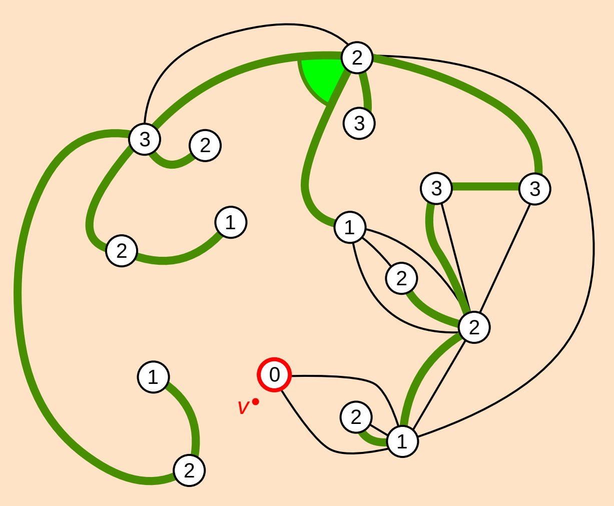

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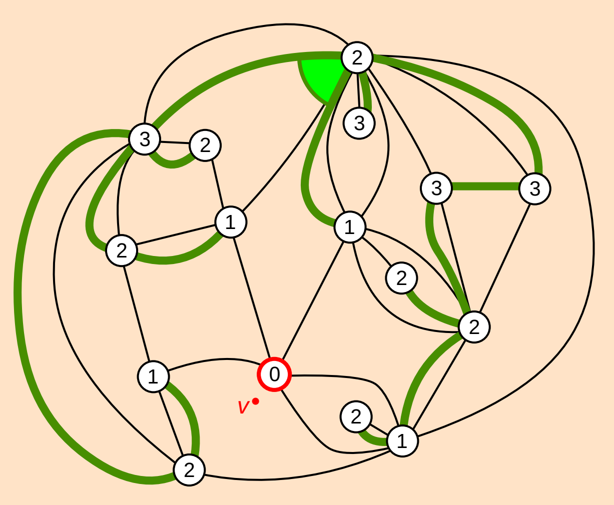

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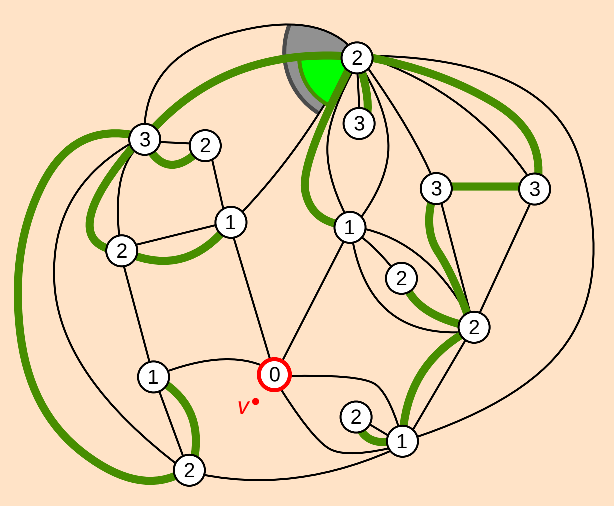
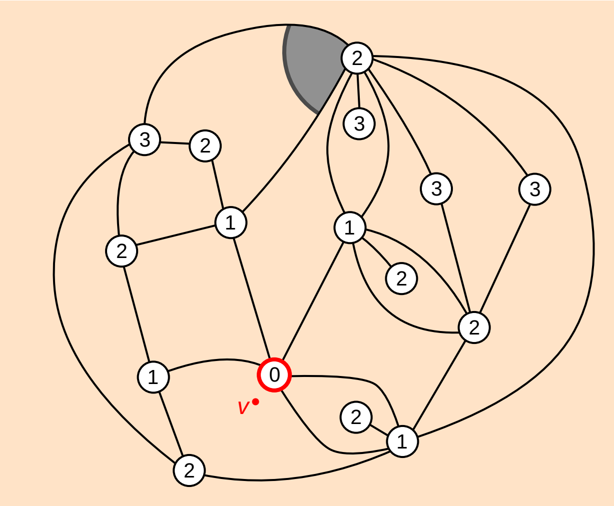


- Take a well-labeled unicellular map.
- Add a vertex $v^\bullet$ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.
- Root and remove the initial edges.

Inverse construction



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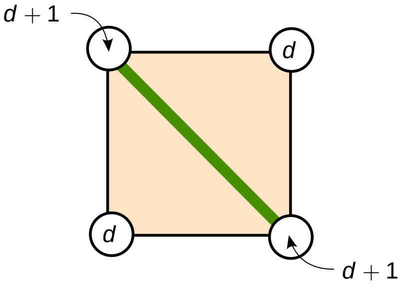
- Link every corner to the first subsequent corner having a strictly smaller label.
- Root and remove the initial edges.

What could go wrong with nonorientable maps?

From quadrangulations to unicellular maps



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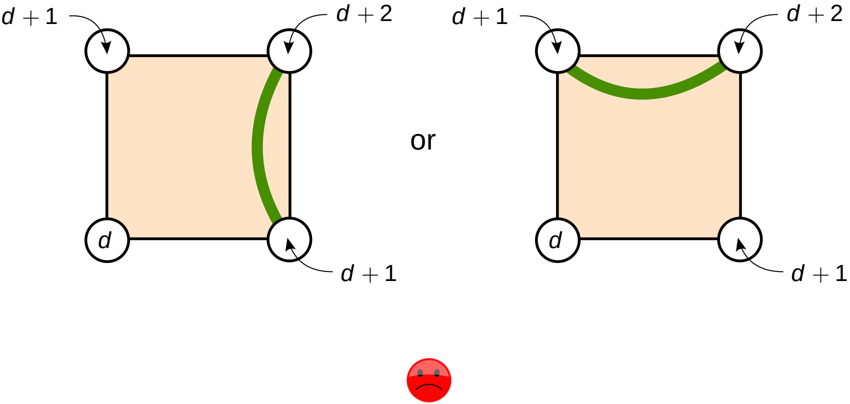

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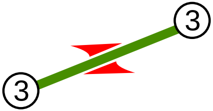

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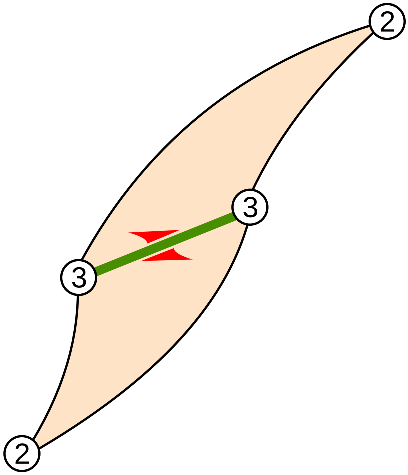

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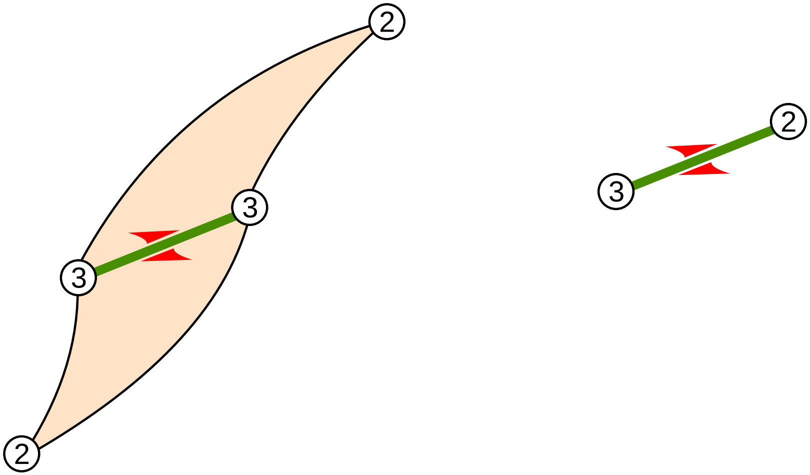

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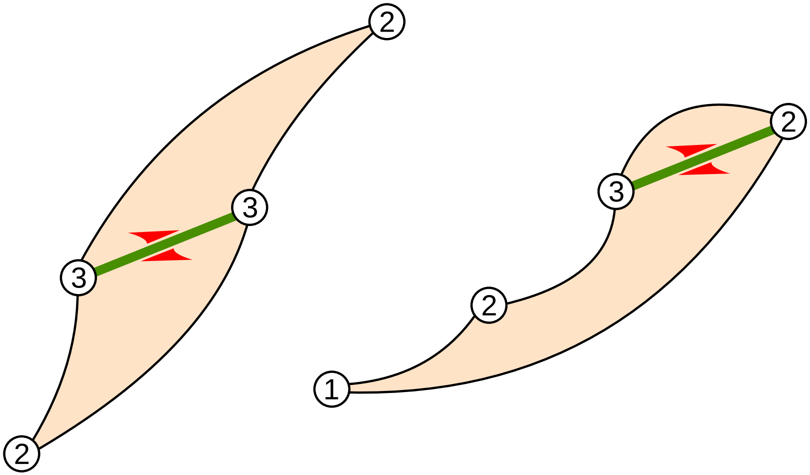

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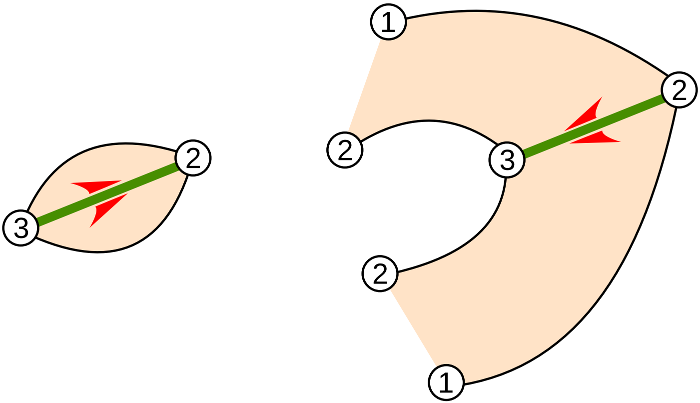
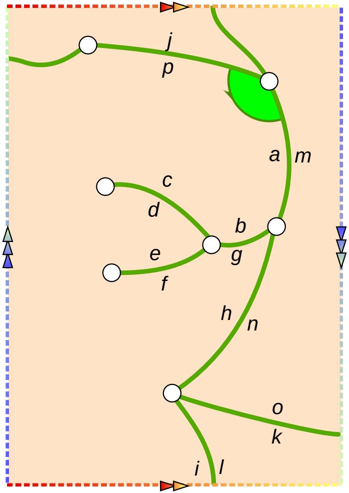
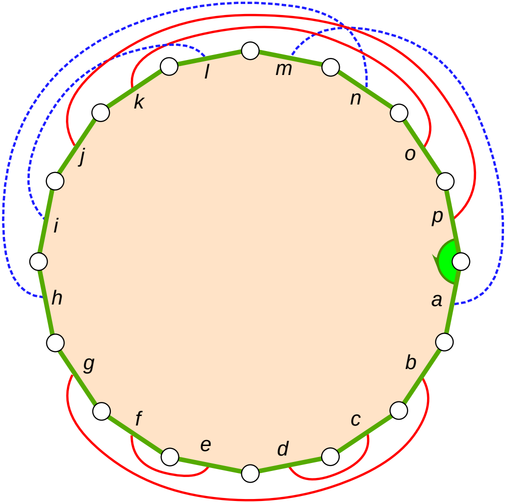
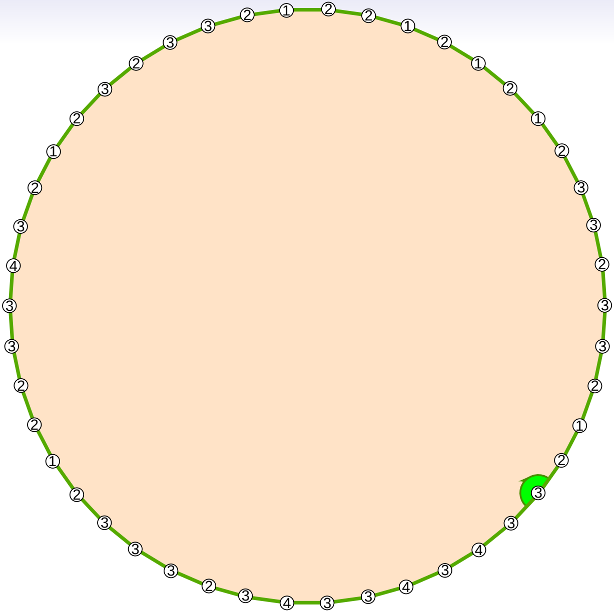


Unicellular maps seen as polygons with paired sides



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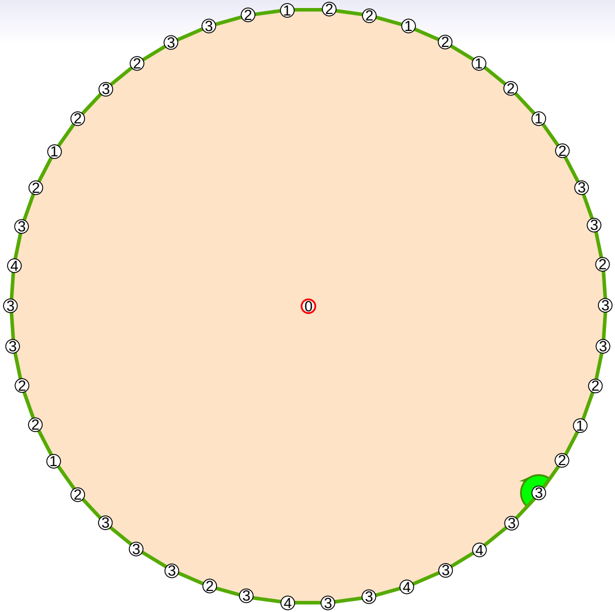

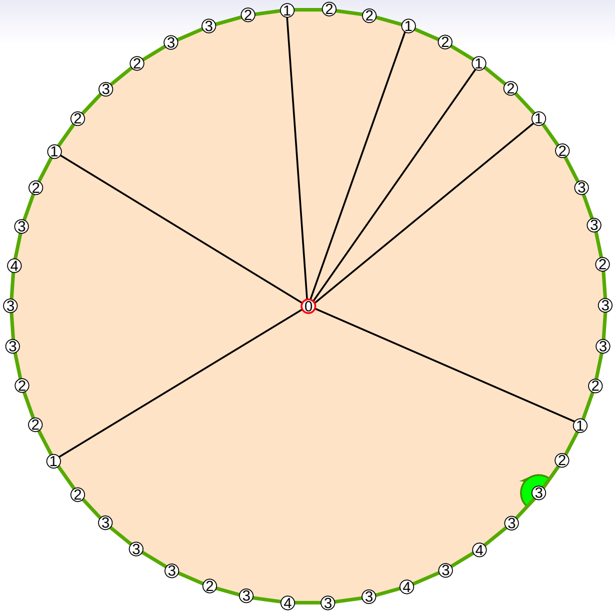
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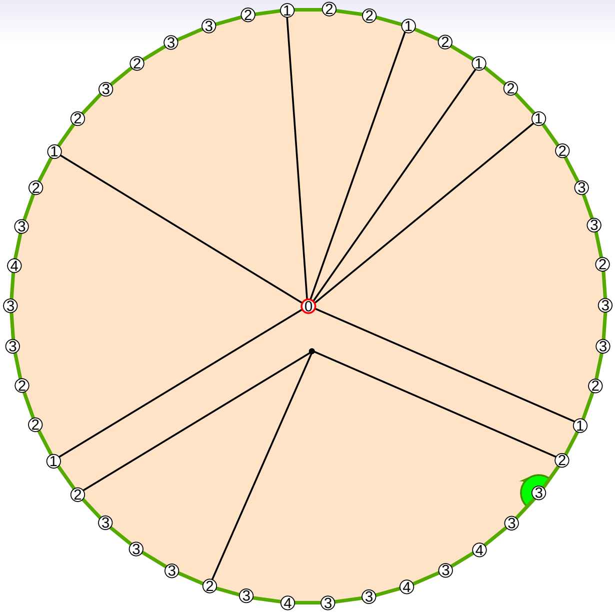
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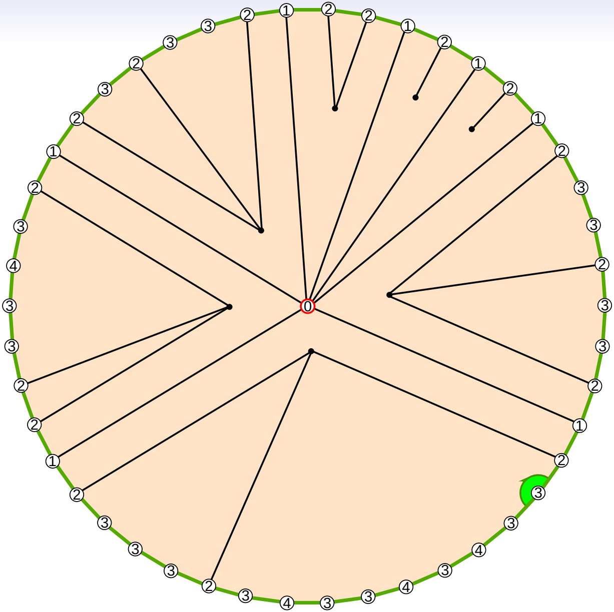
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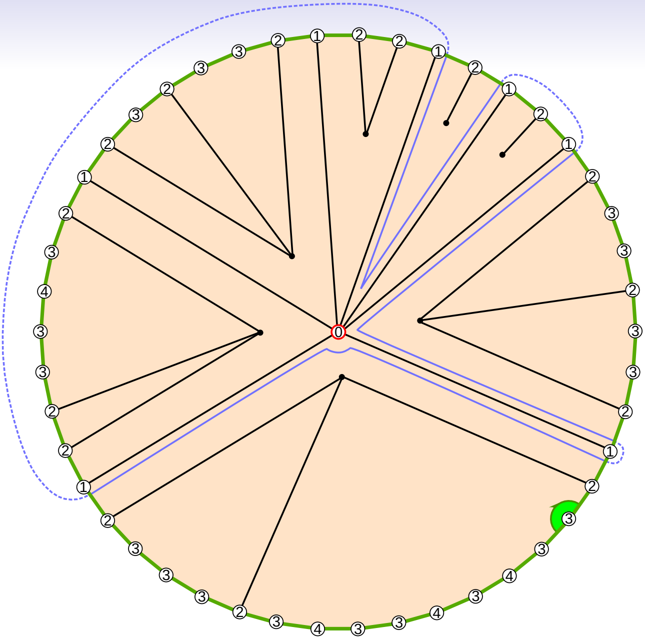

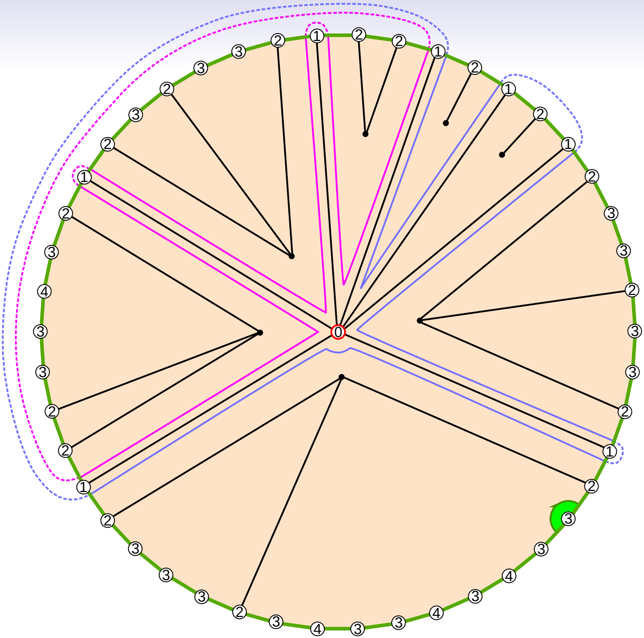

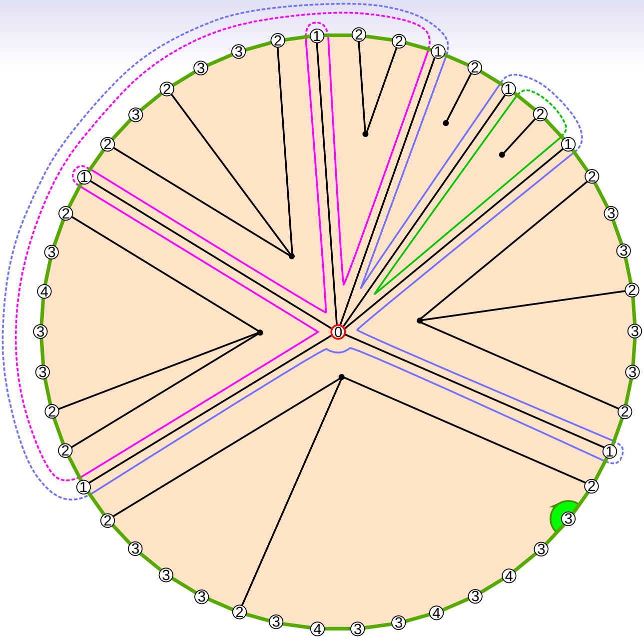
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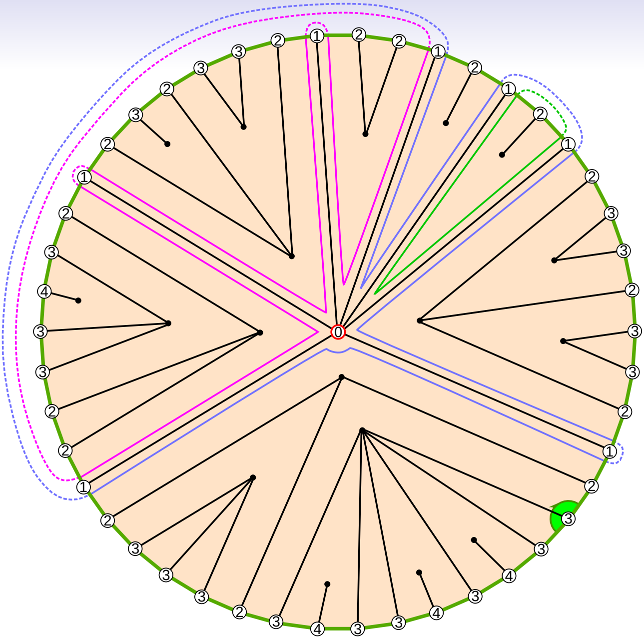

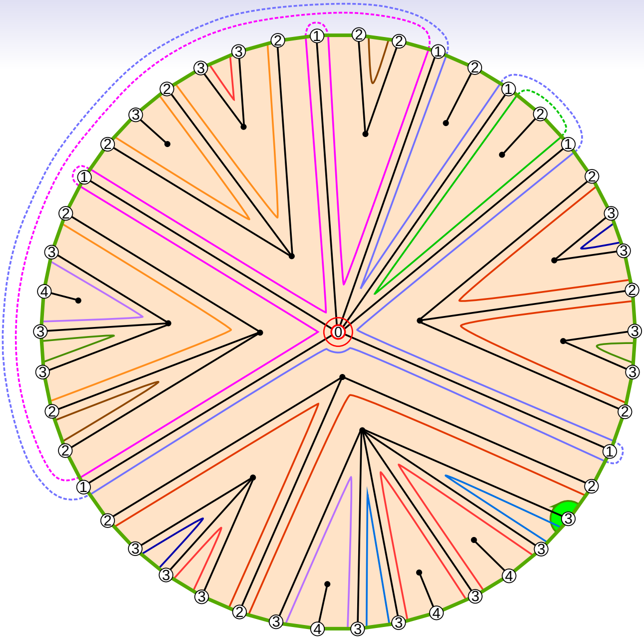

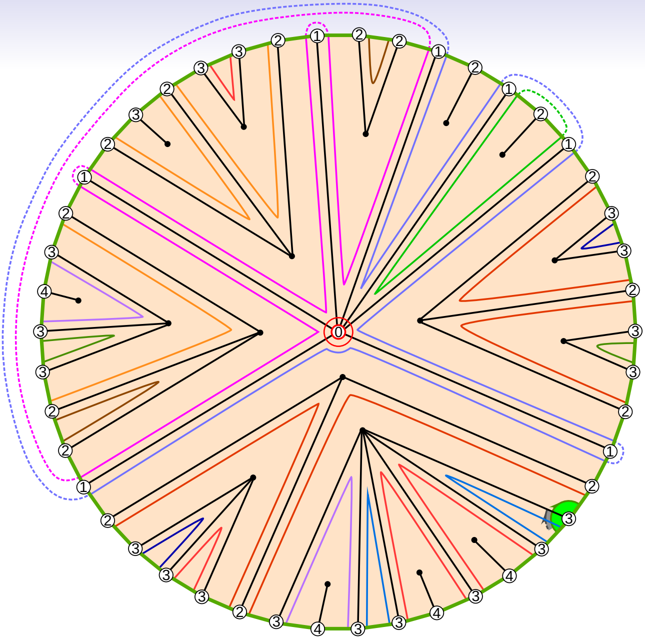

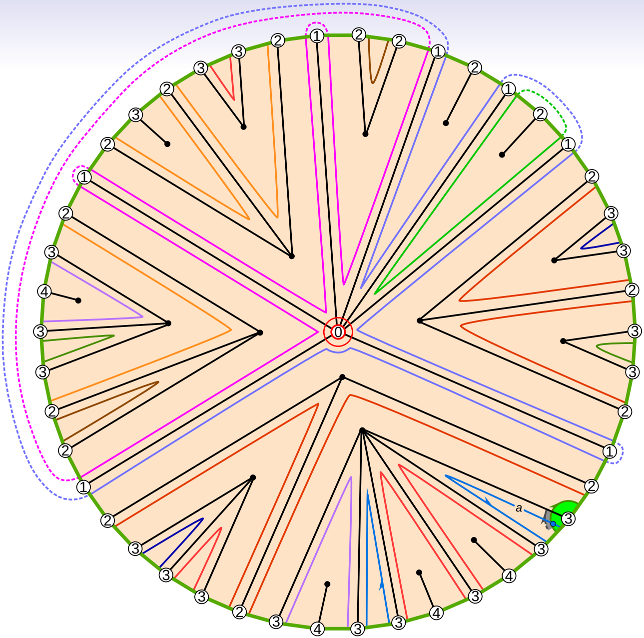
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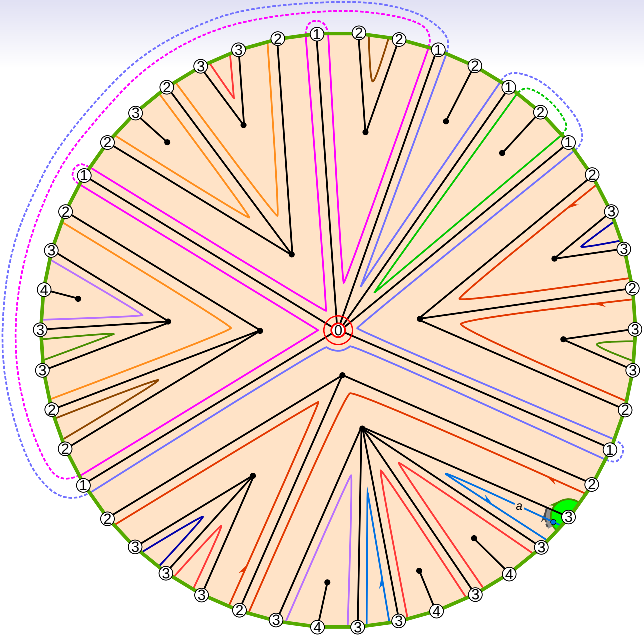
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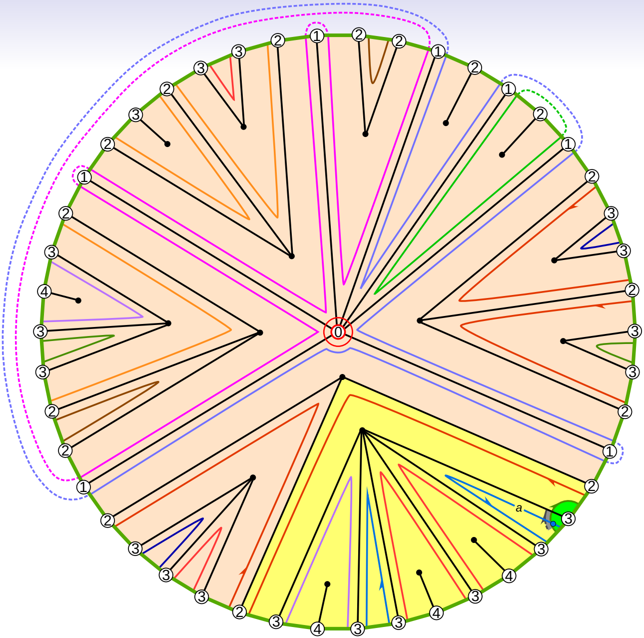

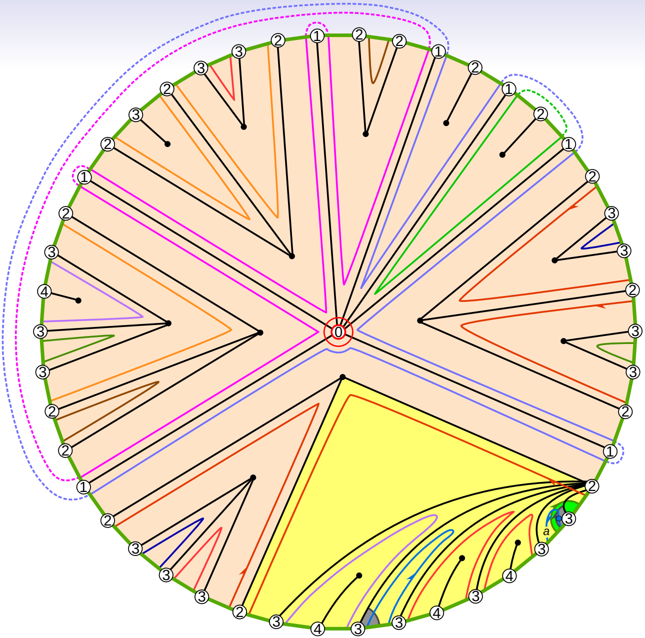
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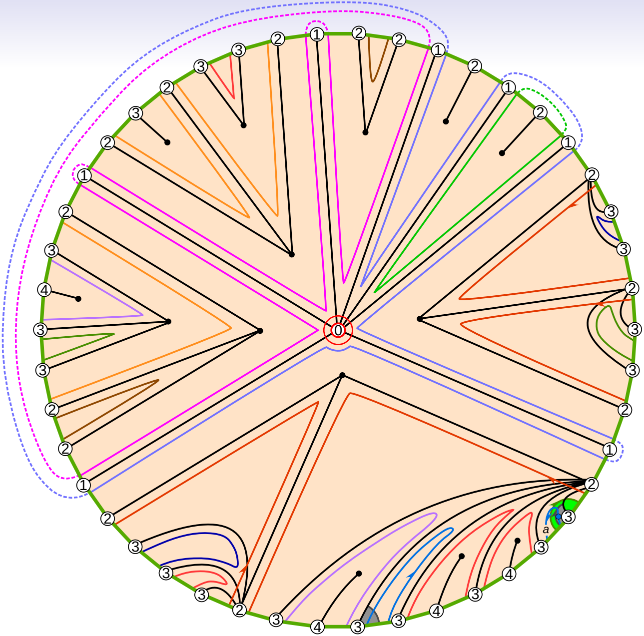
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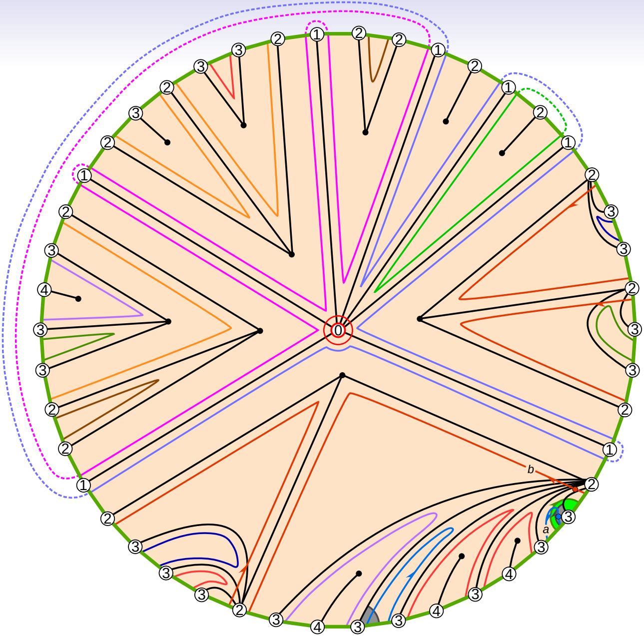
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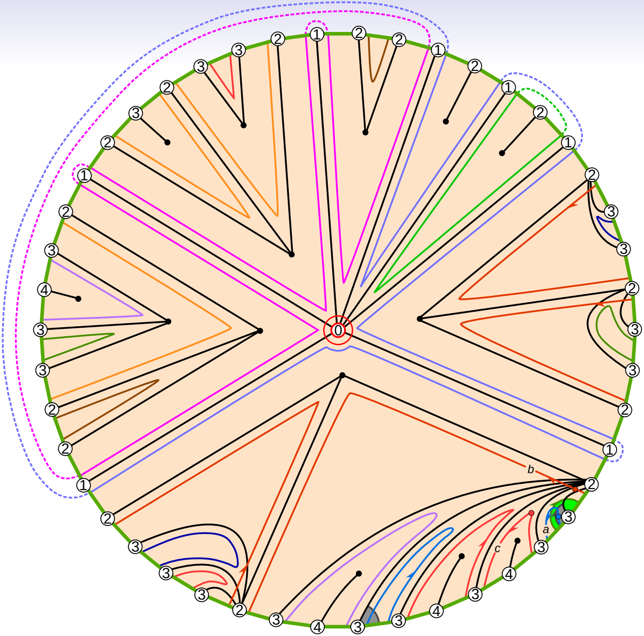
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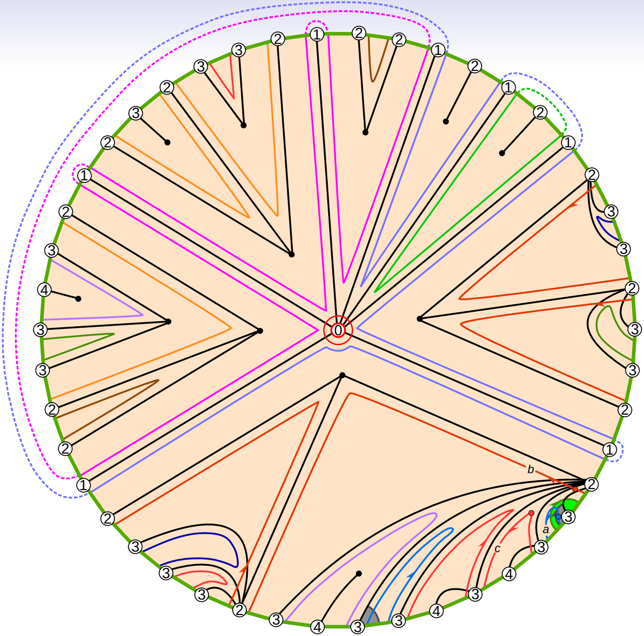
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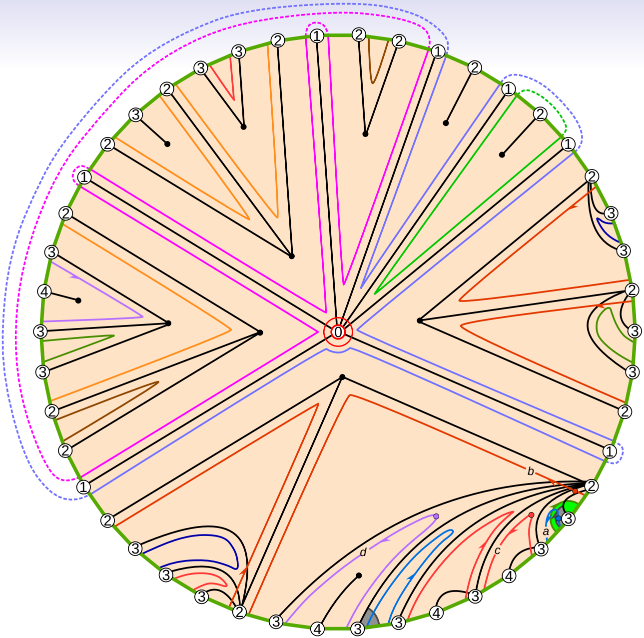
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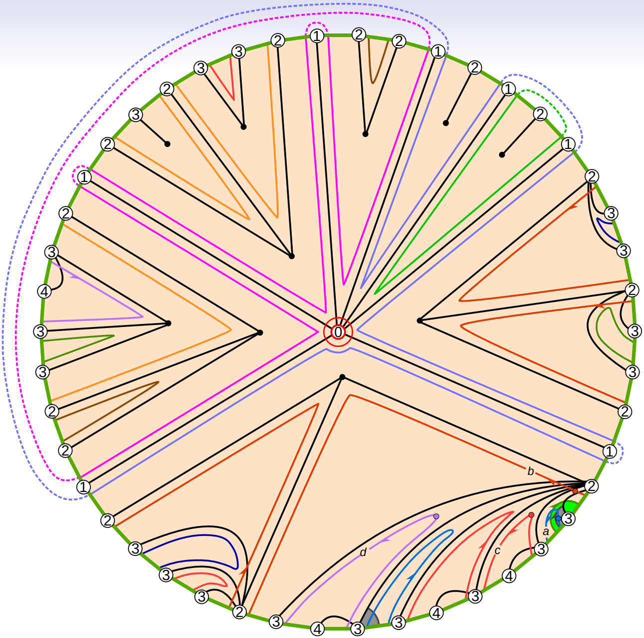
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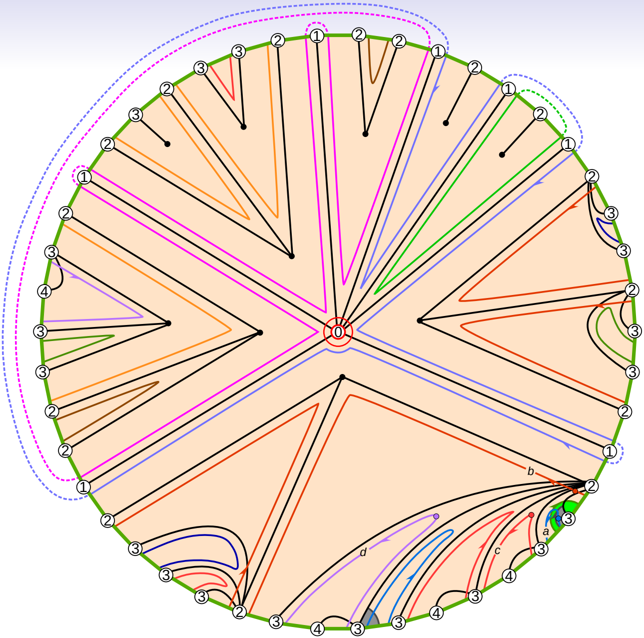
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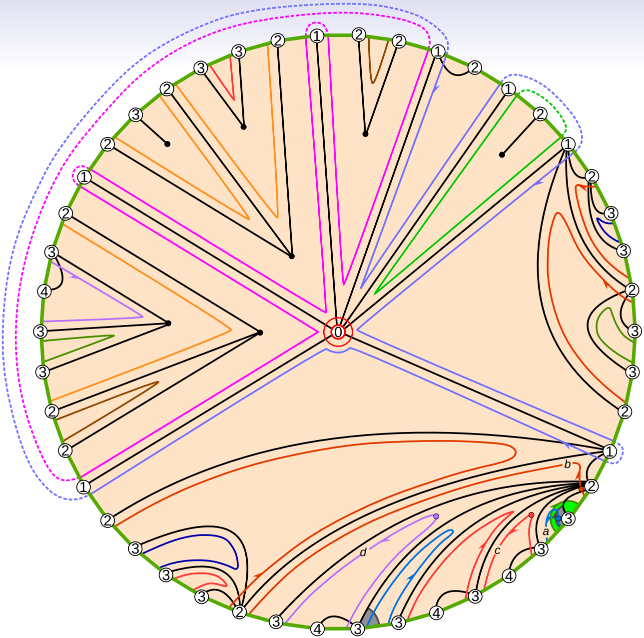
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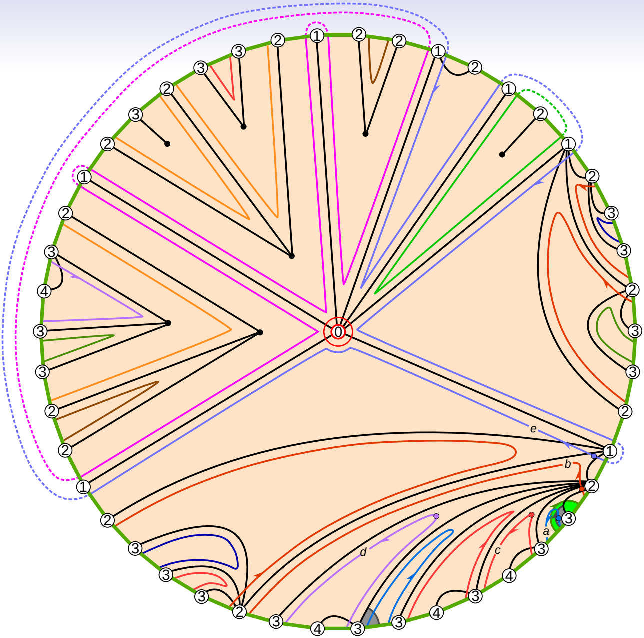
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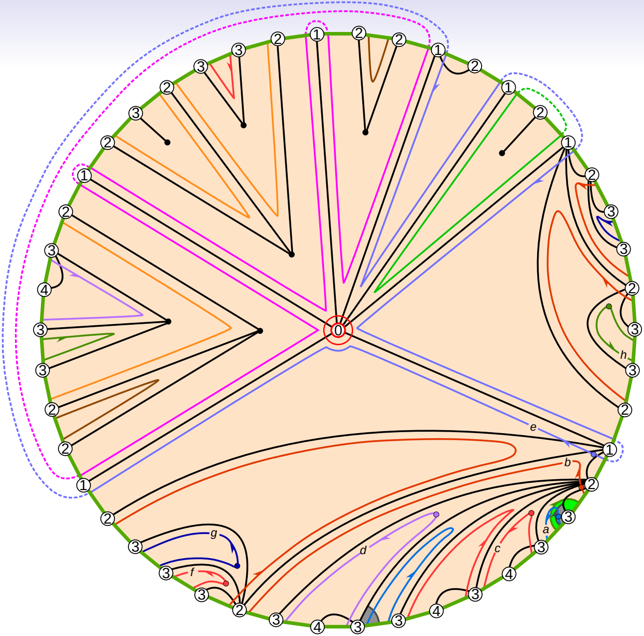
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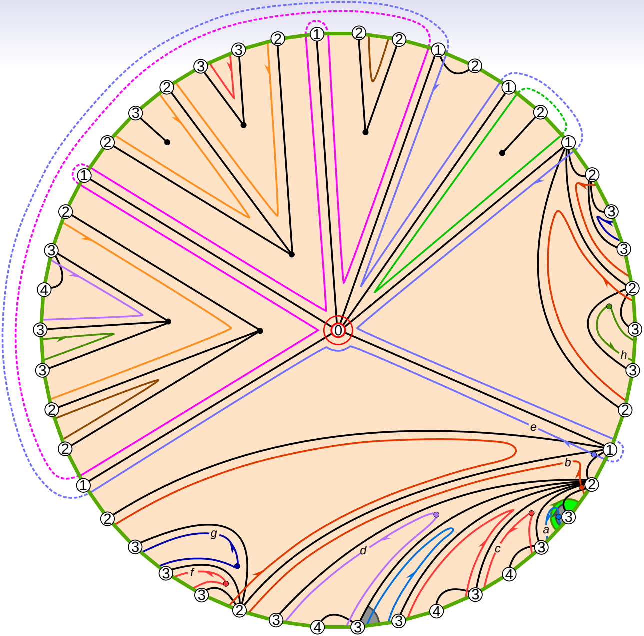

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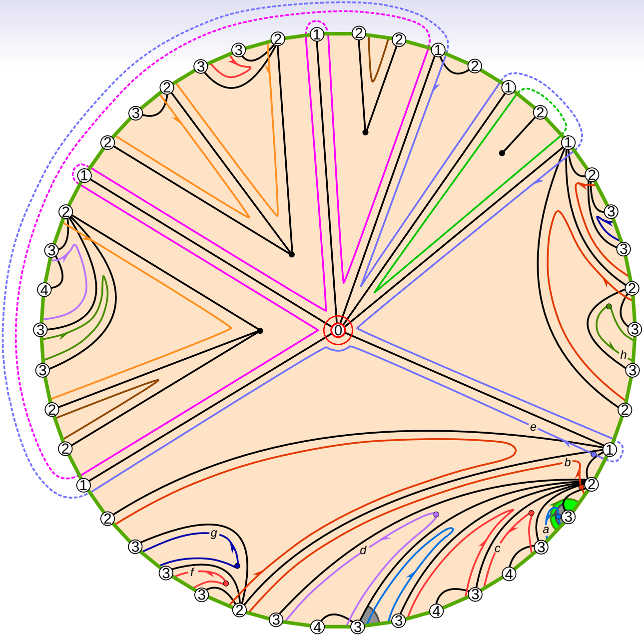
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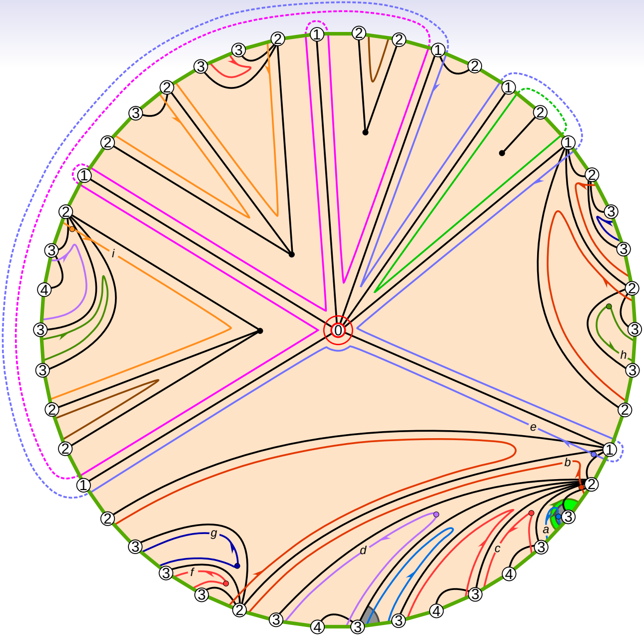
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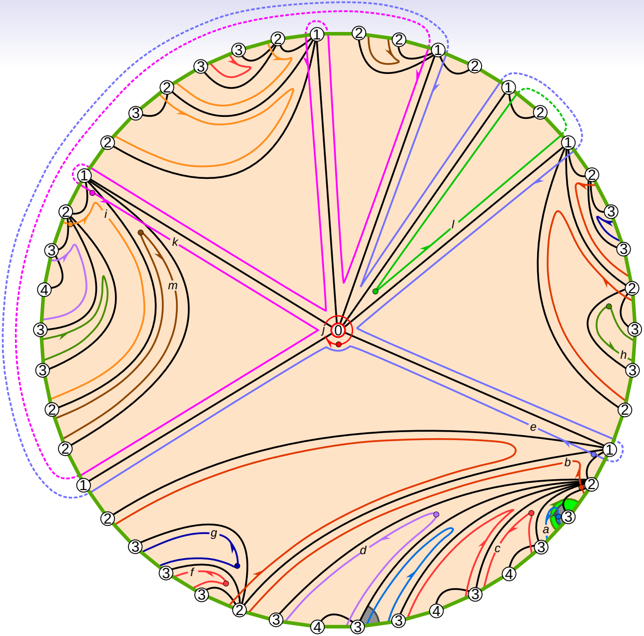
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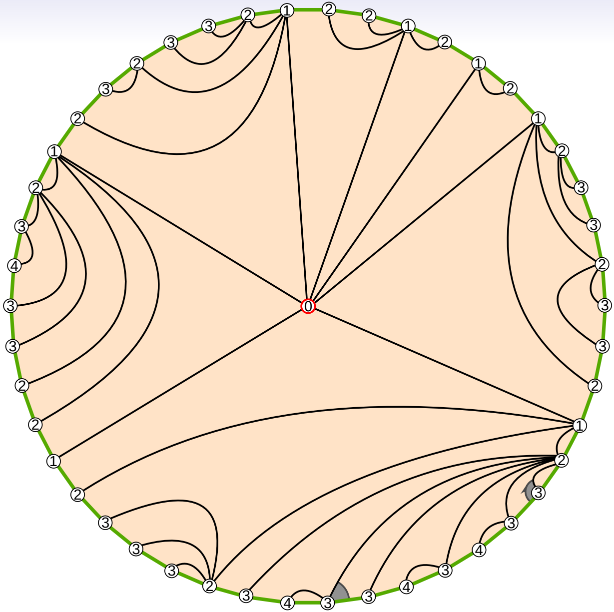
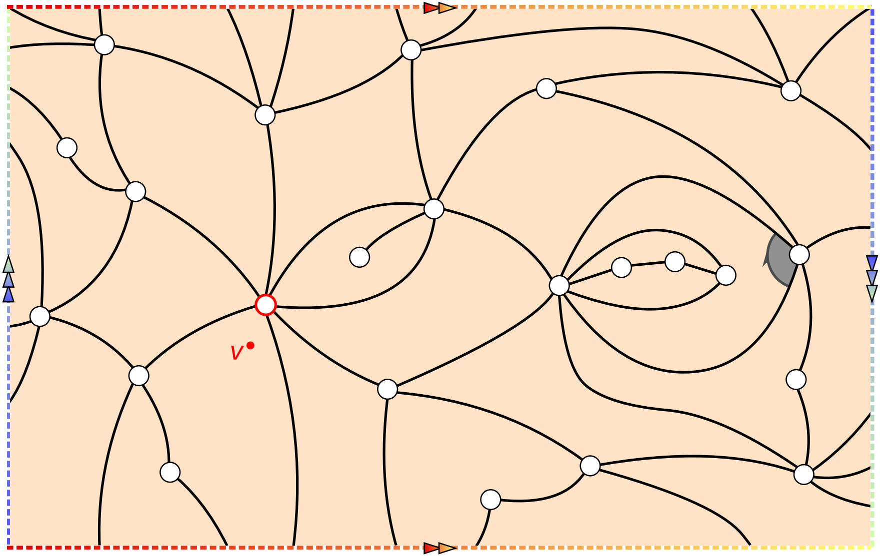


Chapuy–Dołęga (revisited)



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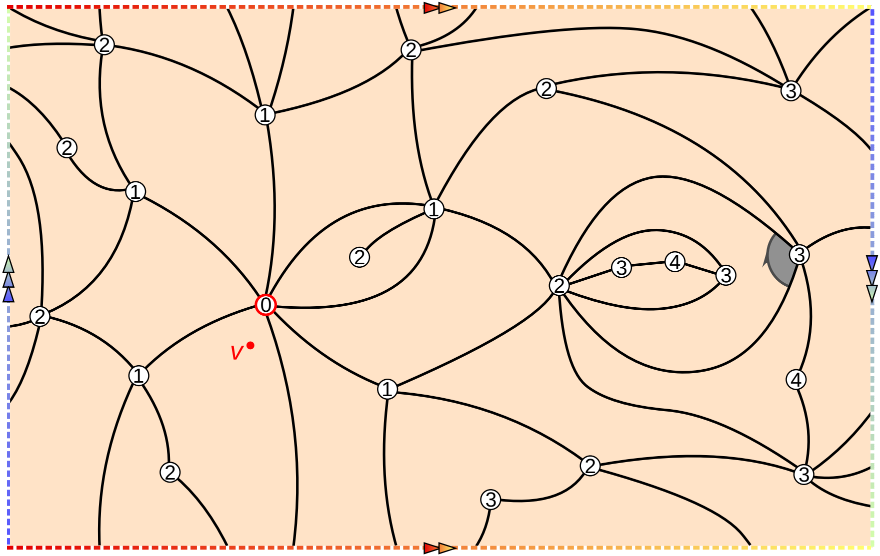

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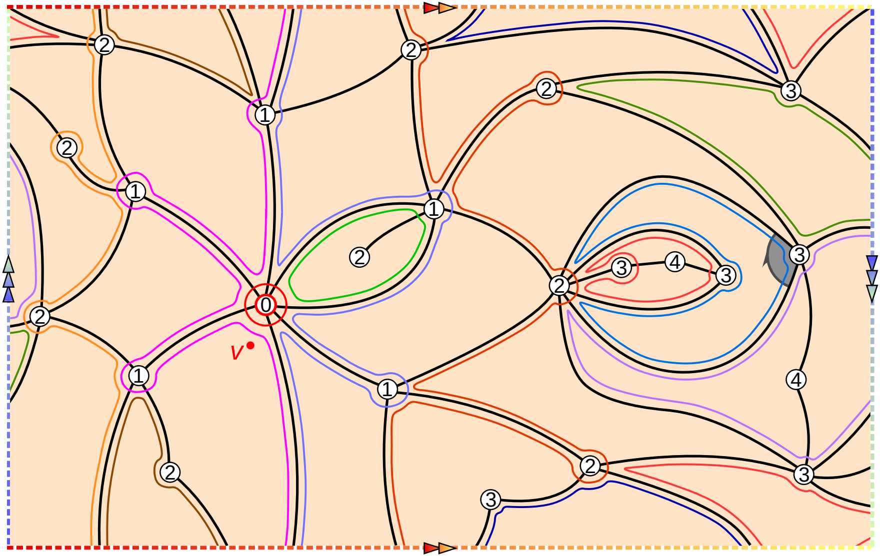

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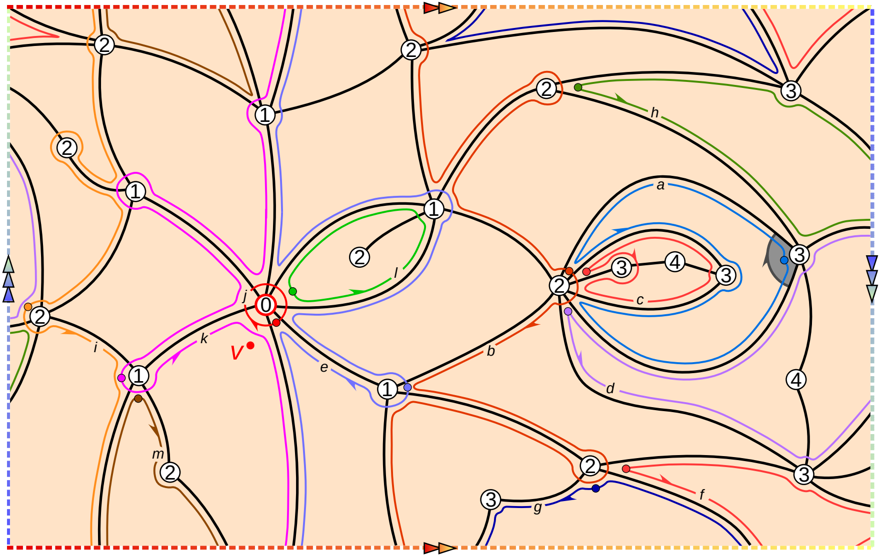

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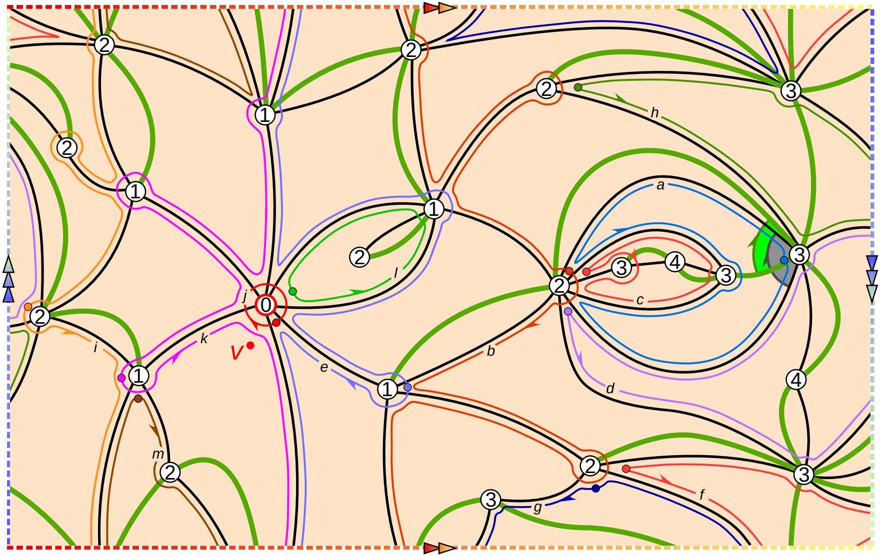

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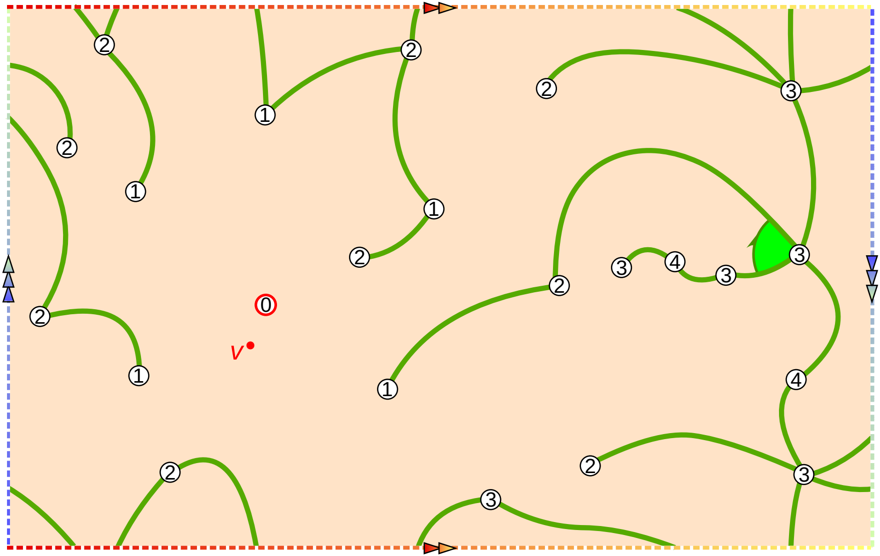
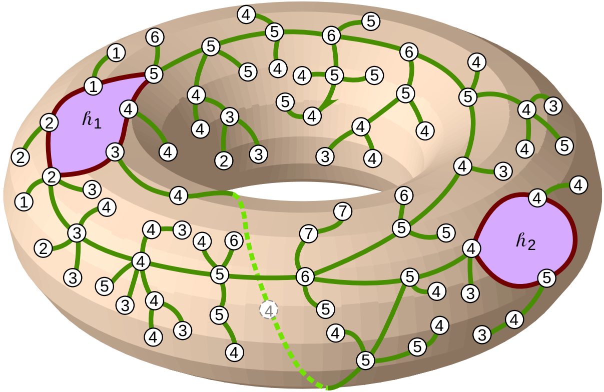
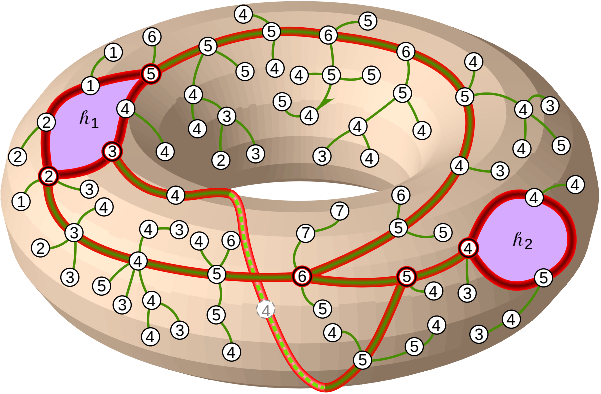
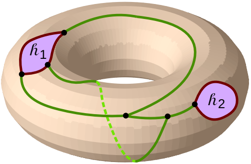


Decomposition into scheme, bridges and forests



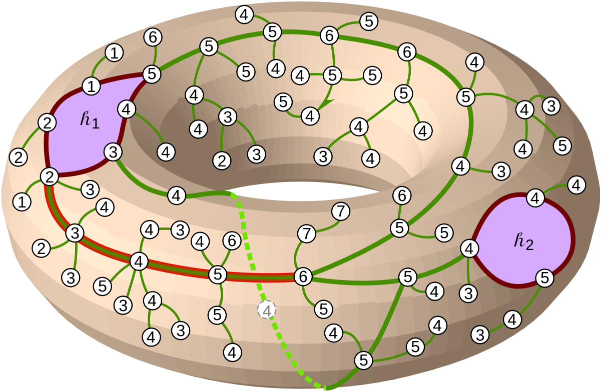
Decomposition into scheme, bridges and forests

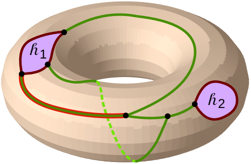




scheme

Decomposition into scheme, bridges and forests

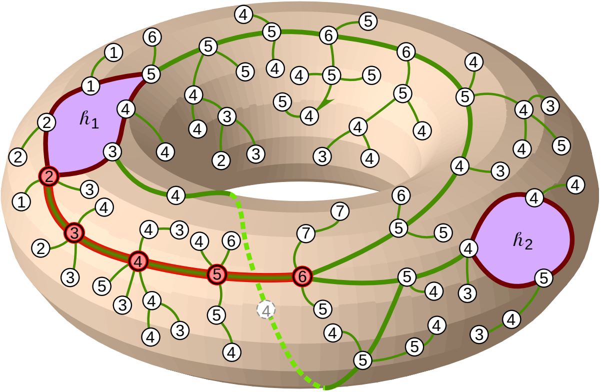


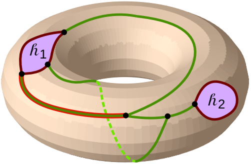


scheme

With each edge of the scheme, we associate:

Decomposition into scheme, bridges and forests



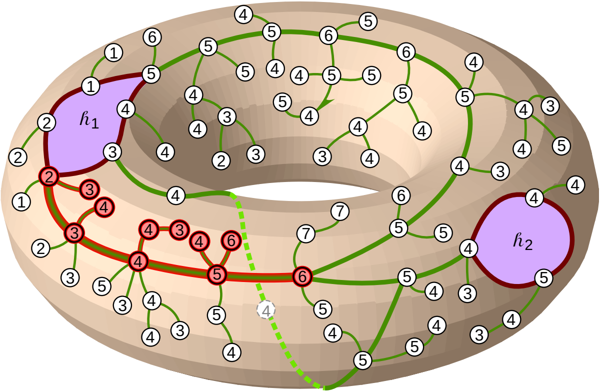


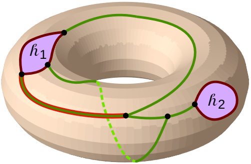
scheme

With each edge of the scheme, we associate:

- a Motzkin bridge

Decomposition into scheme, bridges and forests





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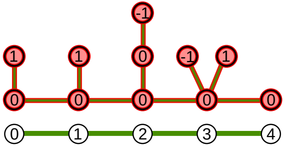
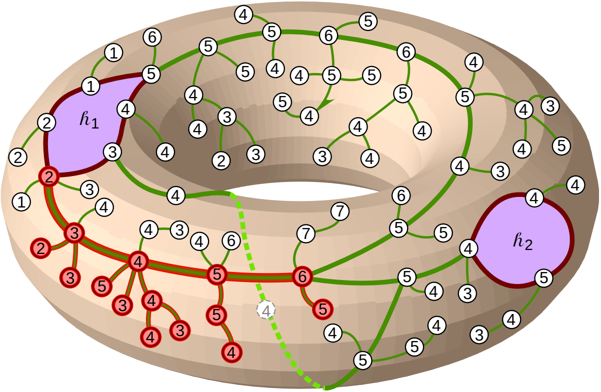
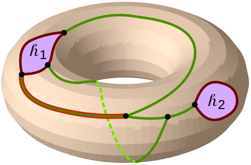


With each edge of the scheme, we associate:

- a Motzkin bridge
- one or two well-labeled forests

Decomposition into scheme, bridges and forests





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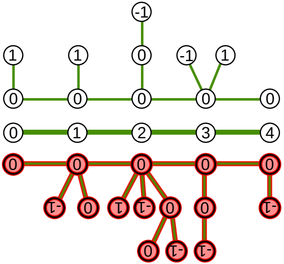

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- a Motzkin bridge
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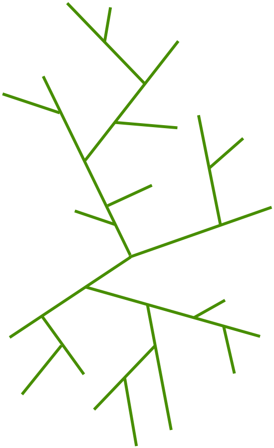
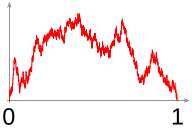
Construction of the Brownian sphere $((g, p) = (0, 0))$

Recall how the Brownian sphere is constructed.



- Consider the CRT, that is, the random real tree encoded by the normalized Brownian excursion.



Construction of the Brownian sphere ($(g, p) = (0, 0)$)

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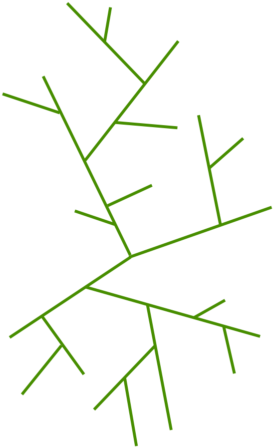

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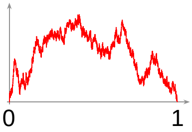

- Put Brownian labels Z on it.

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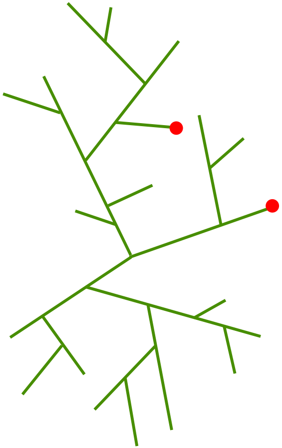

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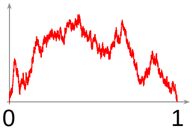

- Put Brownian labels Z on it.
- Identify the points a and b whenever $Z_a = Z_b = \min_{[a,b]} Z$.

Construction of the Brownian sphere ($(g, p) = (0, 0)$)

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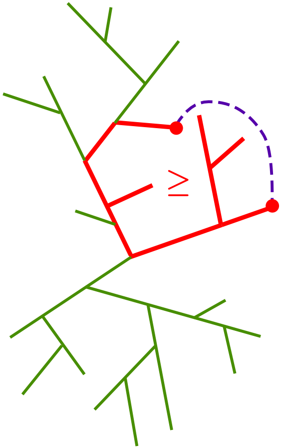

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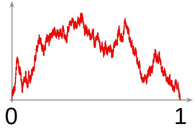

- Put Brownian labels Z on it.
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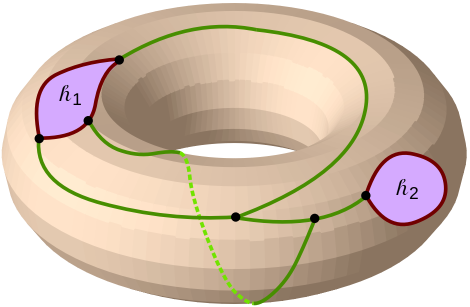
Construction in general ($(g, p) \neq (0, 0)$)

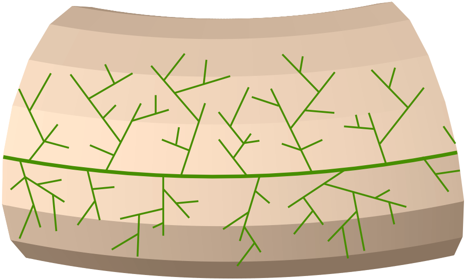
Any Brownian surface may be constructed as follows.



- Start with the proper analog to the CRT:
  it is a dominant scheme with Brownian forests grafted on every edge (except inside the holes).

Construction in general $((g, p) \neq (0, 0))$

Any Brownian surface may be constructed as follows.



- Start with the proper analog to the CRT: it is a dominant scheme with Brownian forests grafted on every edge (except inside the holes).

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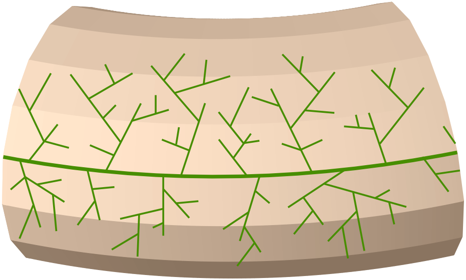

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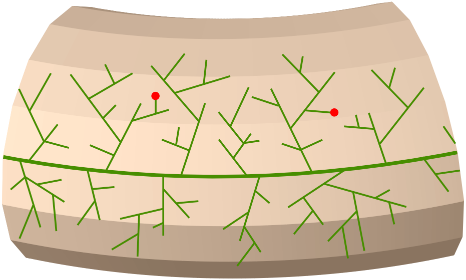

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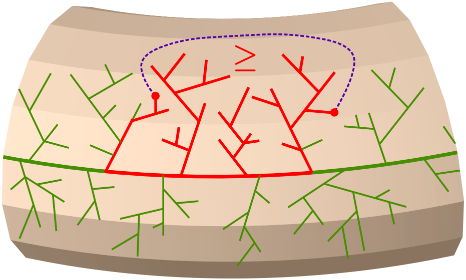

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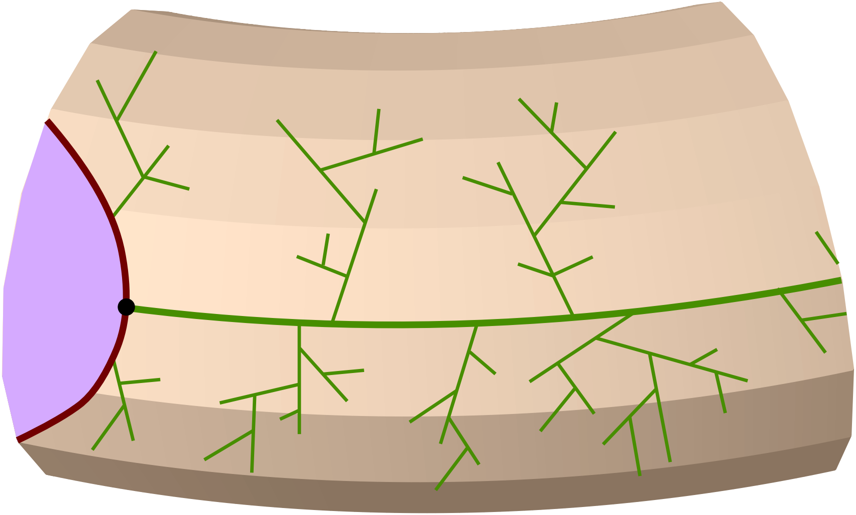

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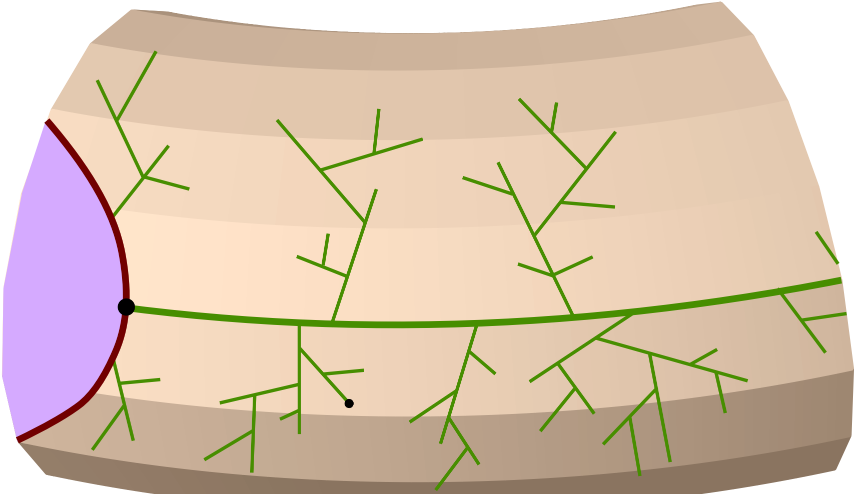

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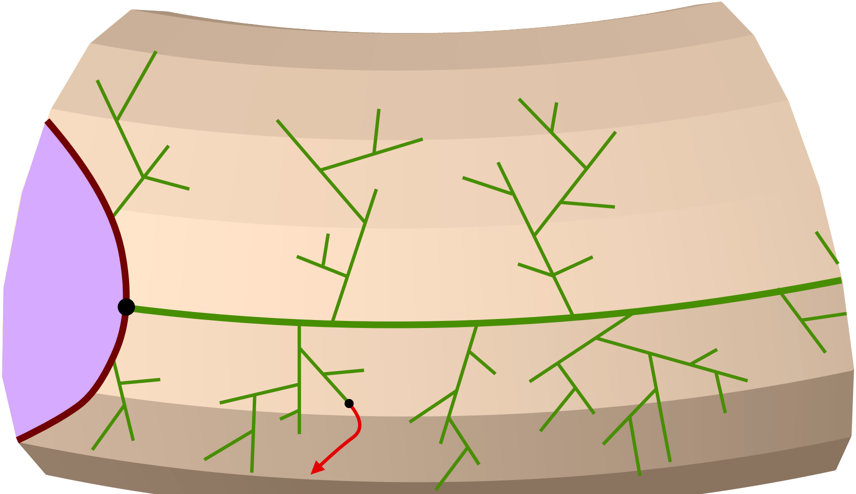

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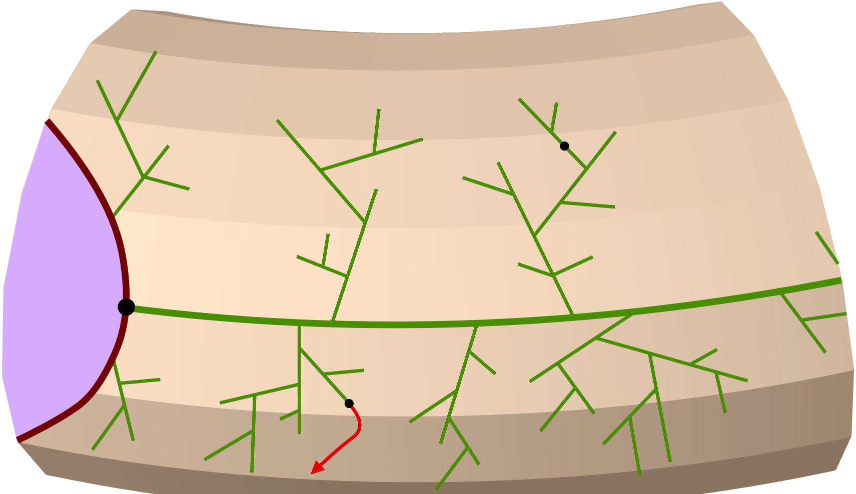

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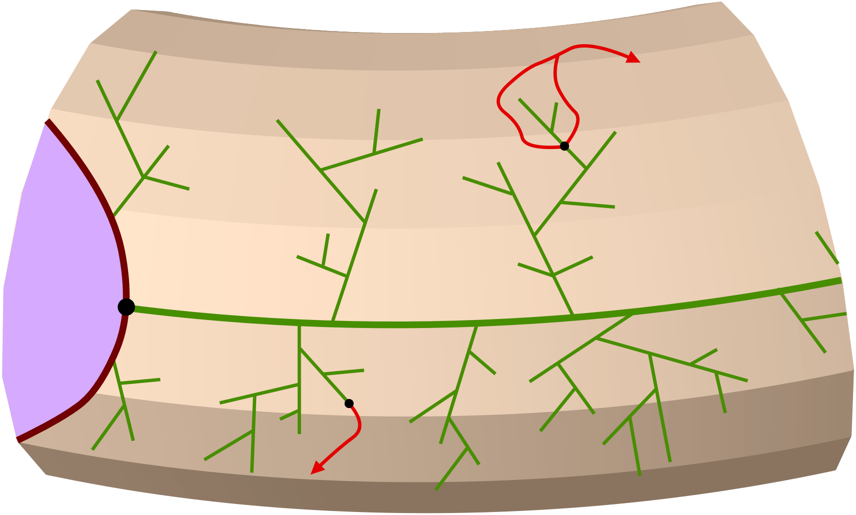

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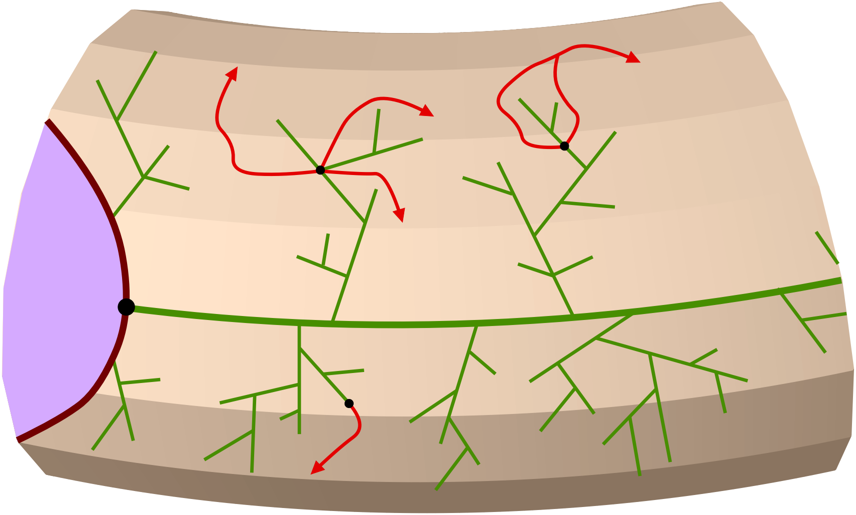

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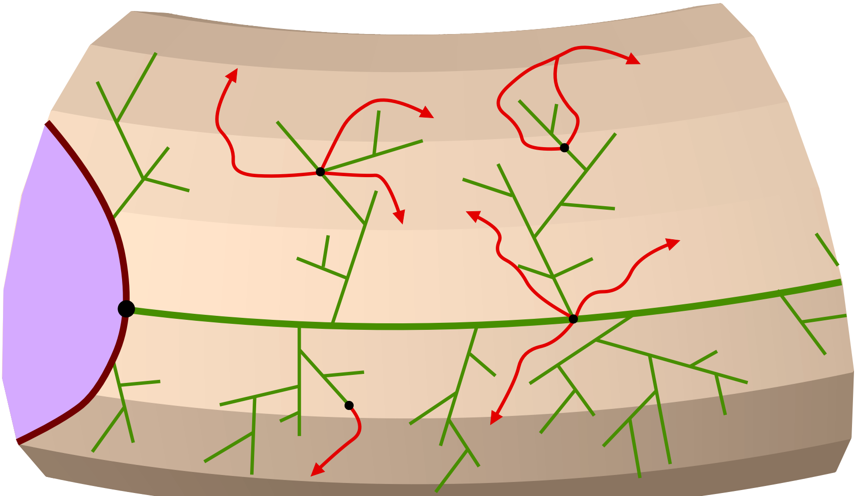

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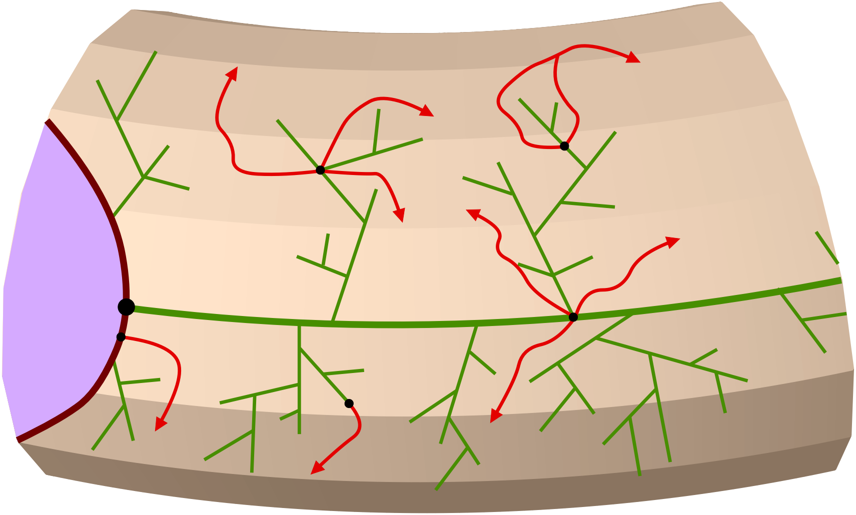

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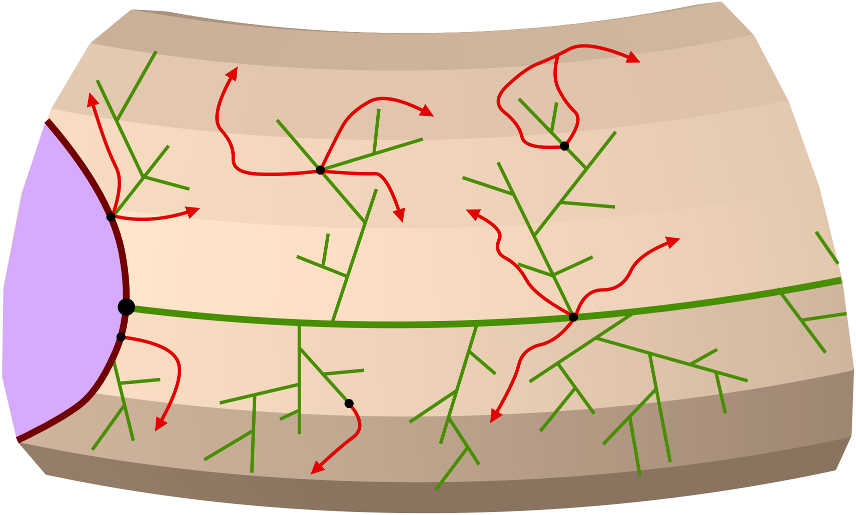

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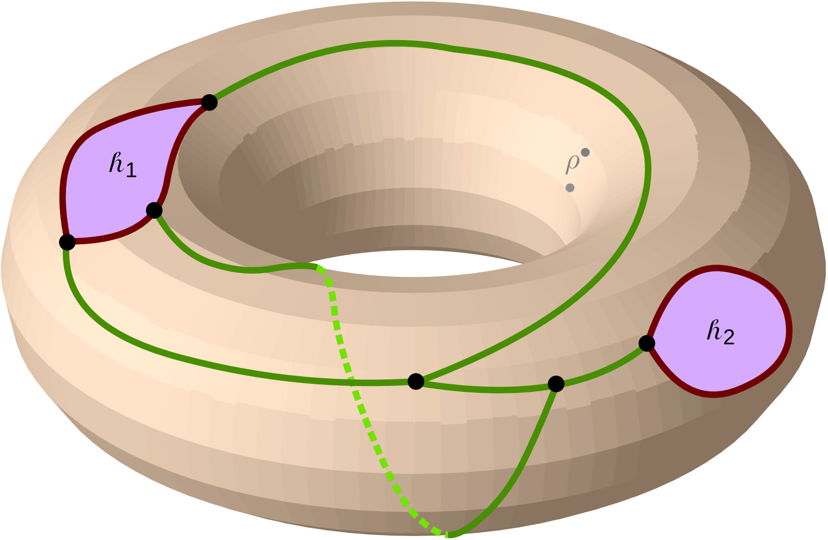

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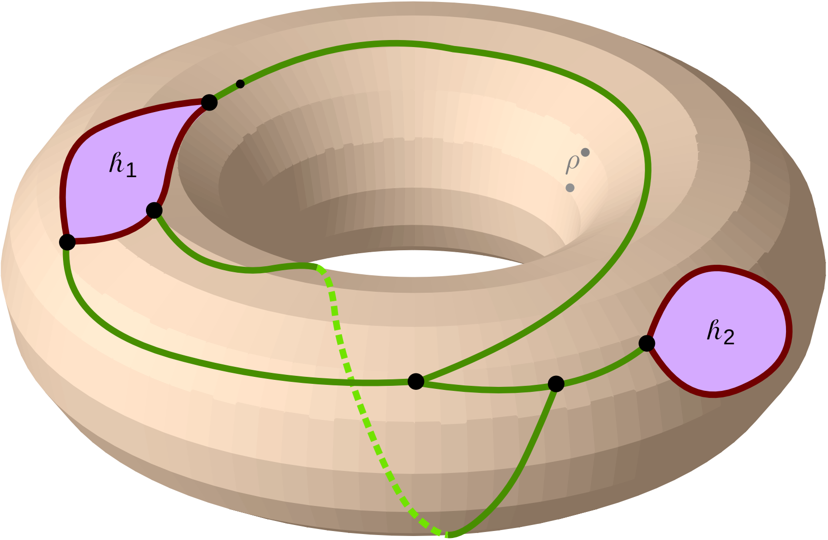

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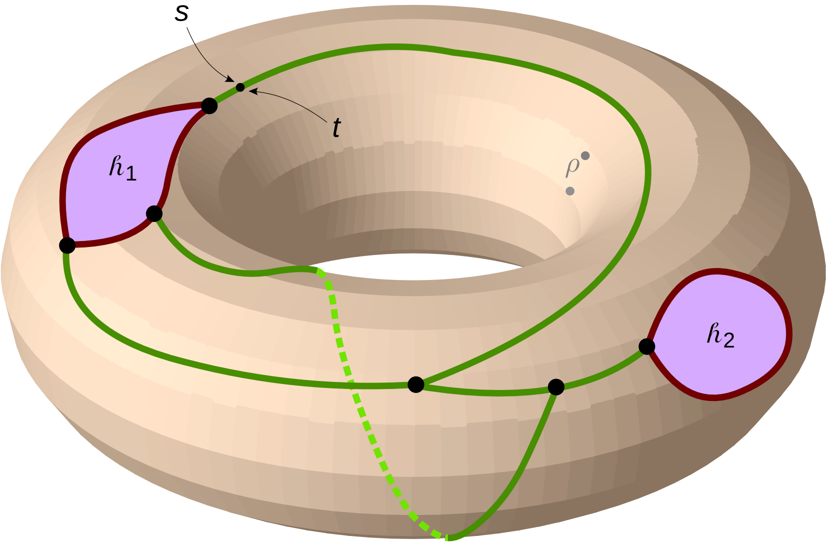

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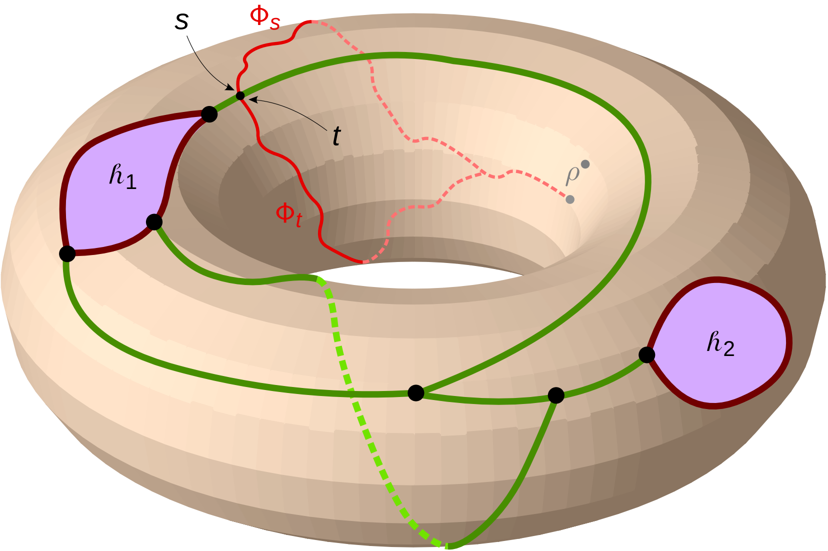

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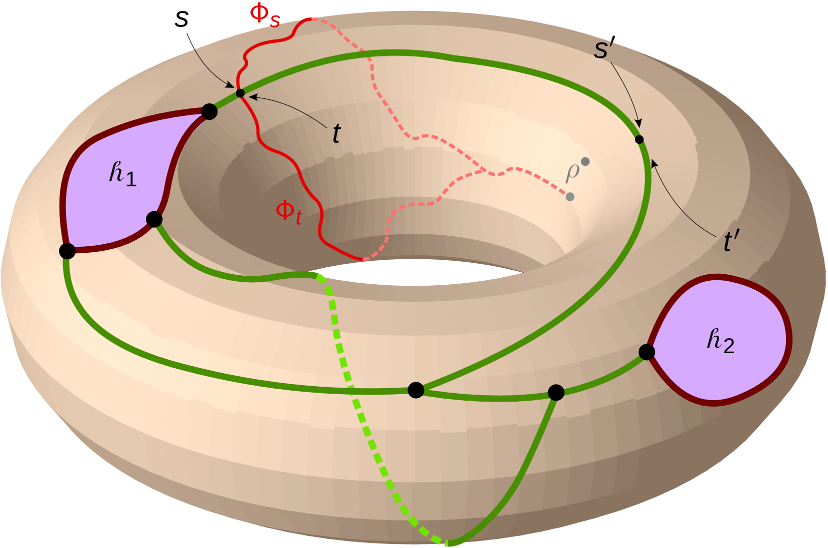

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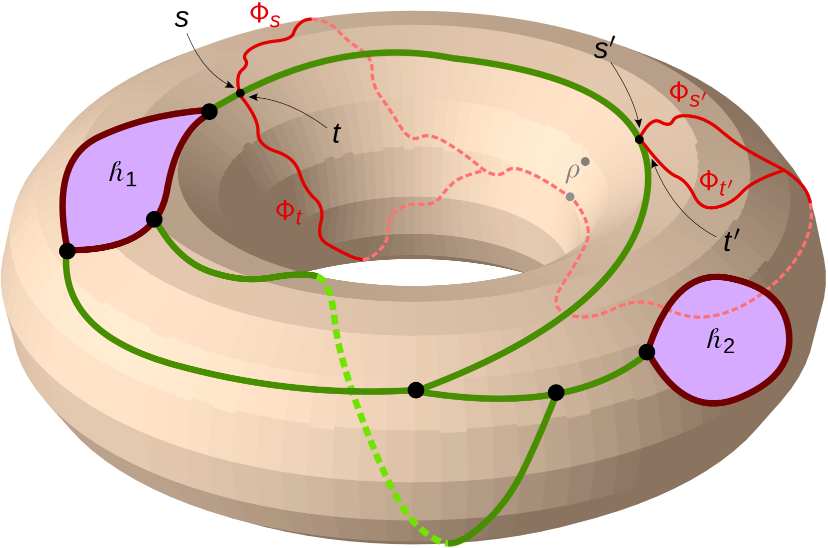

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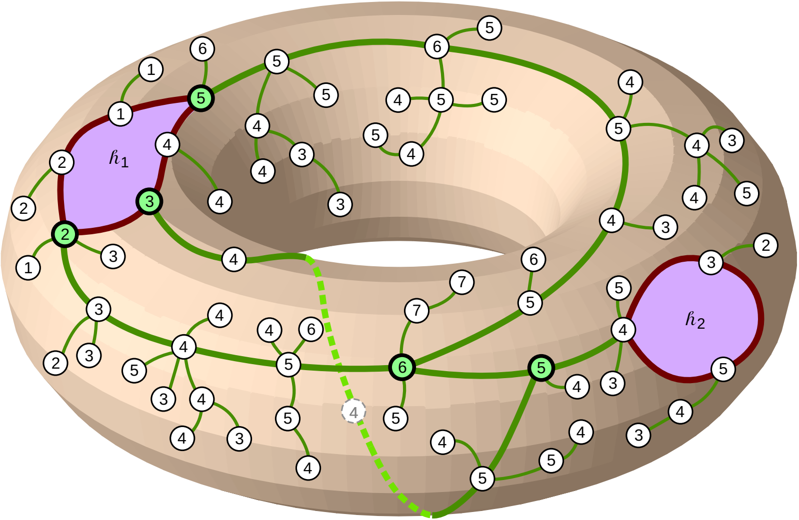

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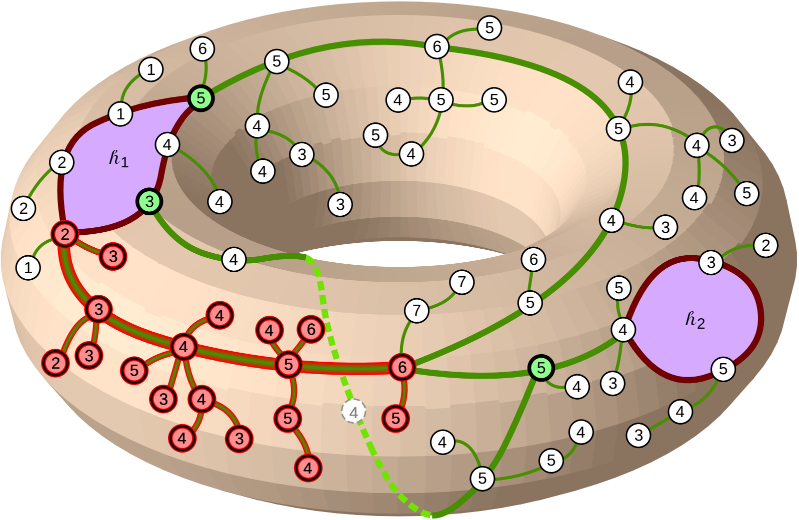

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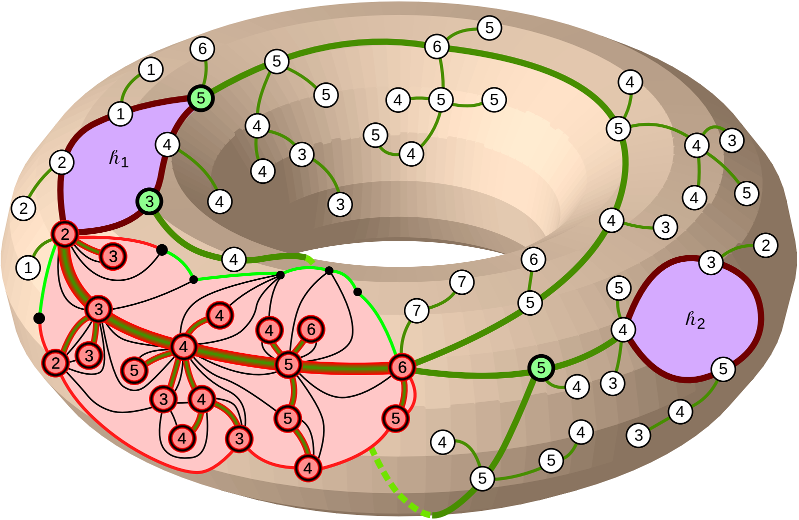

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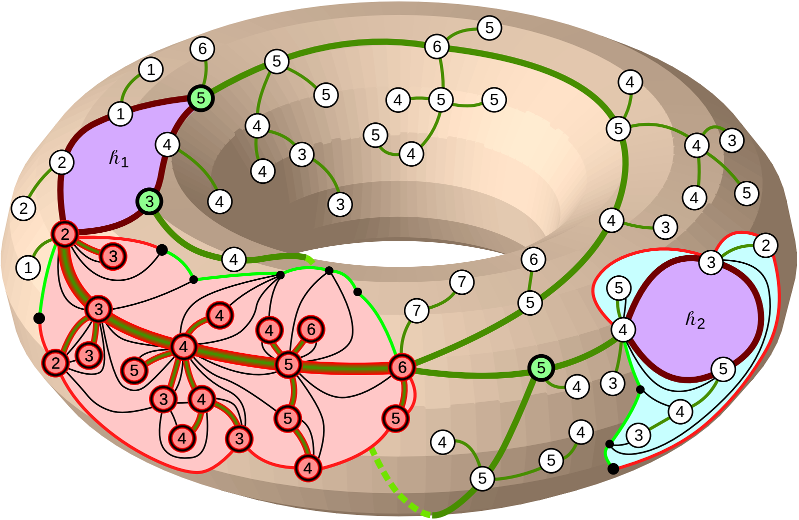

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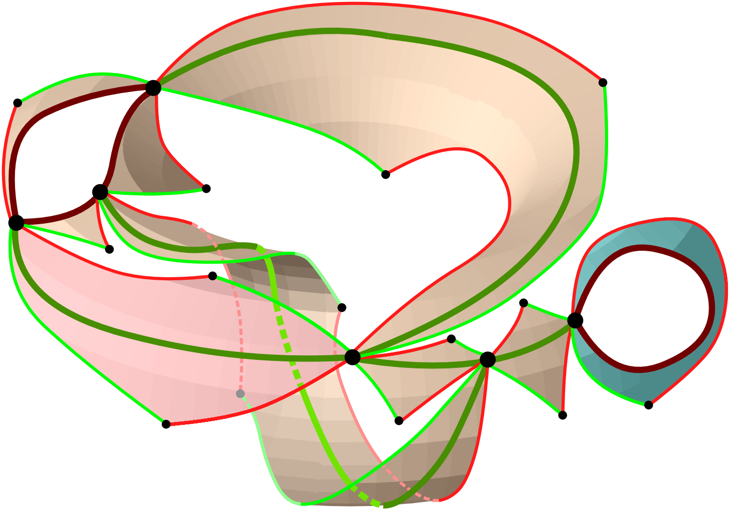

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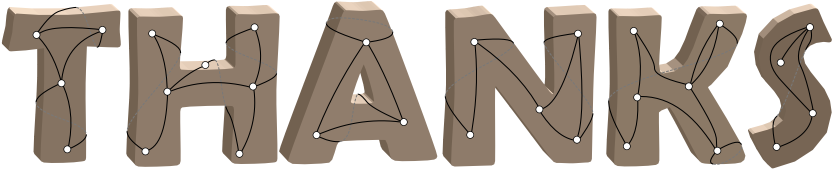
- Glue back everything together.
  - Obtain the uniqueness of the limit.
  - Obtain for free the topology and Hausdorff dimensions.


THANKS

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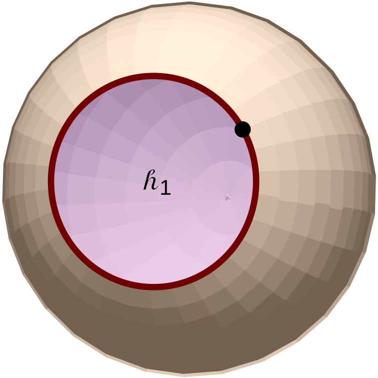

The only scheme for the disk

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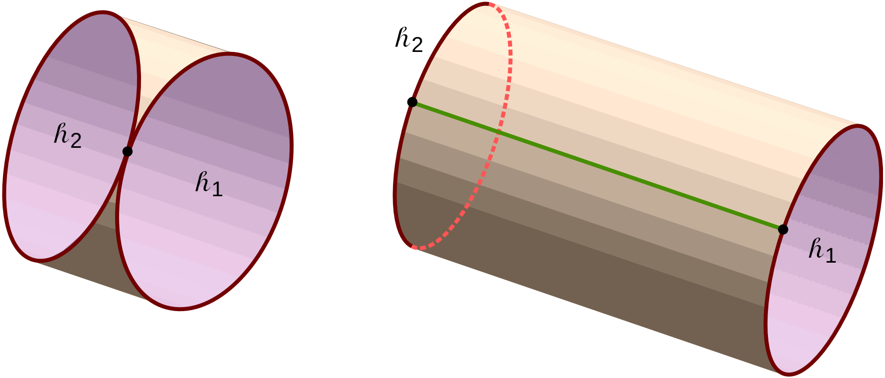

The two cylindrical schemes. Only the one on the right is dominant.

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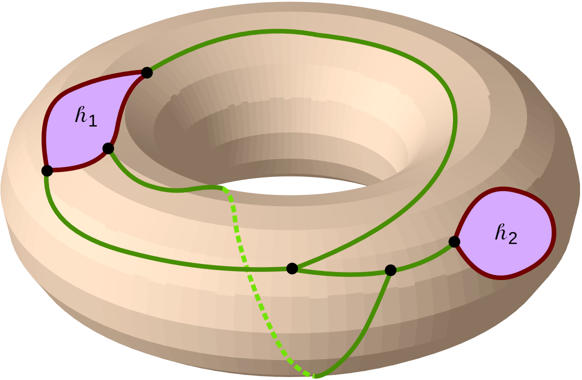

One possible (dominant) scheme for the torus with 2 holes.

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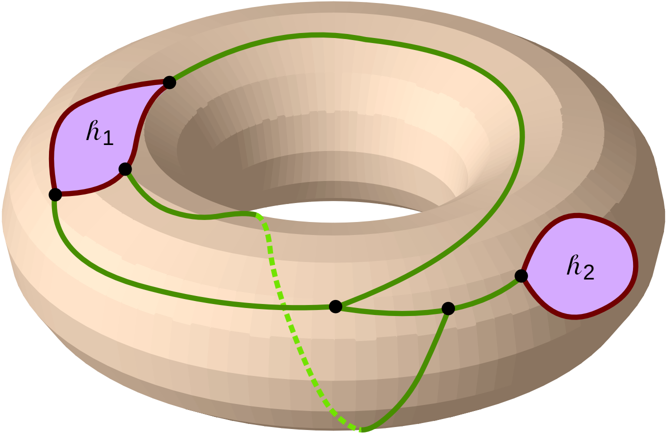

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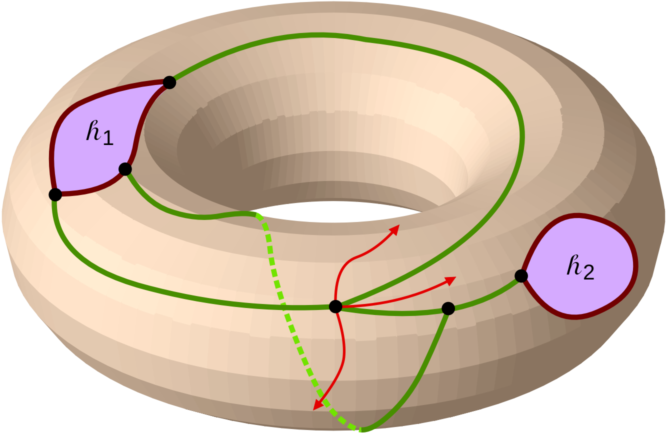

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