A bijection for nonorientable general maps

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Aim

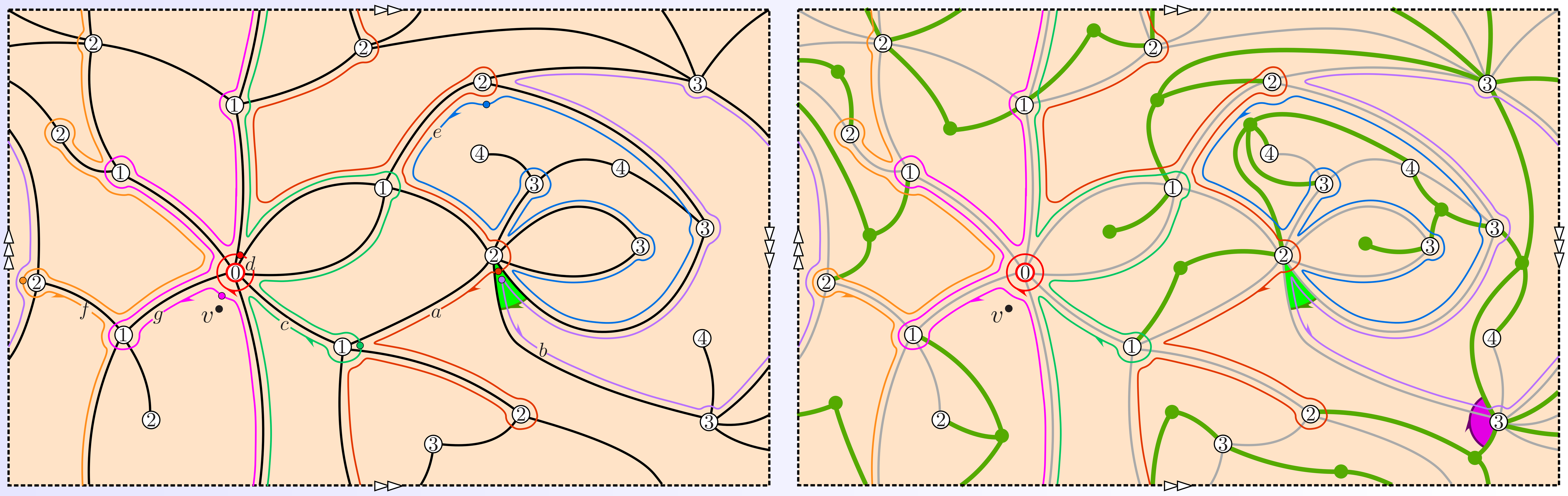
We extend Chapuy–Dołęga's bijection [CD15] for nonorientable bipartite quadrangulations to nonorientable general maps.

Level loops

Our construction uses the concept of *level loops* in a pointed map $(\mathfrak{m}, v^\bullet)$. A level loop of level i is constructed on the faces of $\mathfrak{m}$ by crossing the edges linking vertices at distance i from $v^\bullet$ to vertices at distance $i+1$ from $v^\bullet$ and by following the other edges (as contour lines in topography). For instance, on the figure below, loop a is at level 2.

Encoding bipartite maps by labeled unicellular mobiles

The vertex labels give the distances to $v^\bullet$. The loops are given an origin and oriented by a global algorithm: starting from the root, we follow the loops one by one from their origin and we give an origin and orient every new loop we encounter.



Among the corners visited by a level loop at level i, we only keep those with label i that are immediately preceded by a corner labeled $i-1$. We add an extra green vertex in the middle of every face to which we link in a noncrossing fashion all the corners we kept.

Corresponding quantities

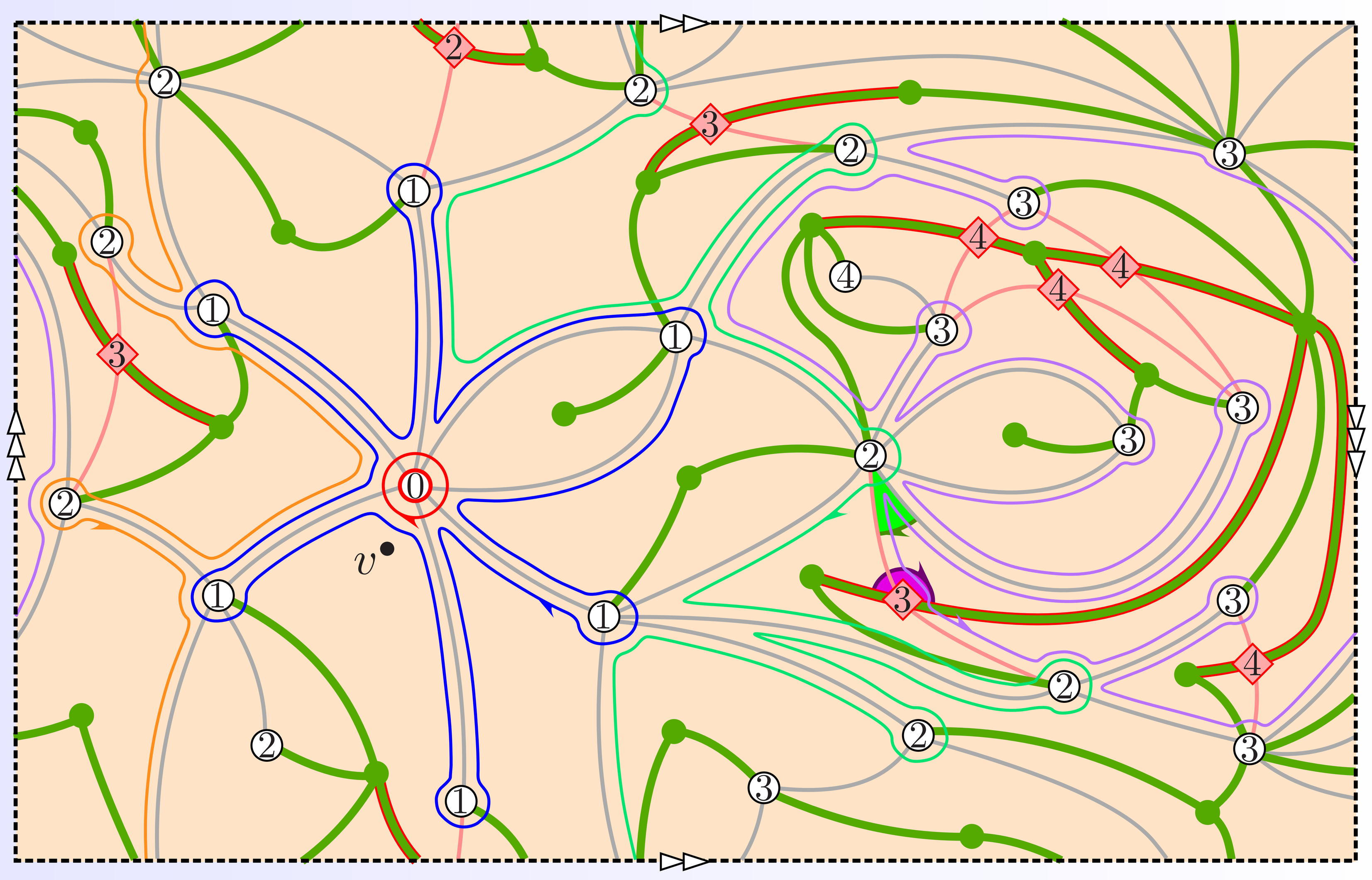
Let $(\mathfrak{m}, v^\bullet)$ be a pointed bipartite map and $\mathfrak{u}$ the associated mobile.

(*i*) $V(\mathfrak{m}) \setminus \{v^\bullet\}$ corresponds to the white vertices of $\mathfrak{u}$ and the label of a white vertex is given by its distance to $v^\bullet$ in $\mathfrak{m}$;

(*ii*) degree $2d$-faces of $\mathfrak{m}$ correspond to degree d-green vertices of $\mathfrak{u}$;

(*iii*) the maps $\mathfrak{m}$ and $\mathfrak{u}$ have the same number of edges.

Encoding general maps



We add an extra vertex on each equilabeled edges. These extra degree 2-vertices remain degree 2-vertices in the encoding mobile.

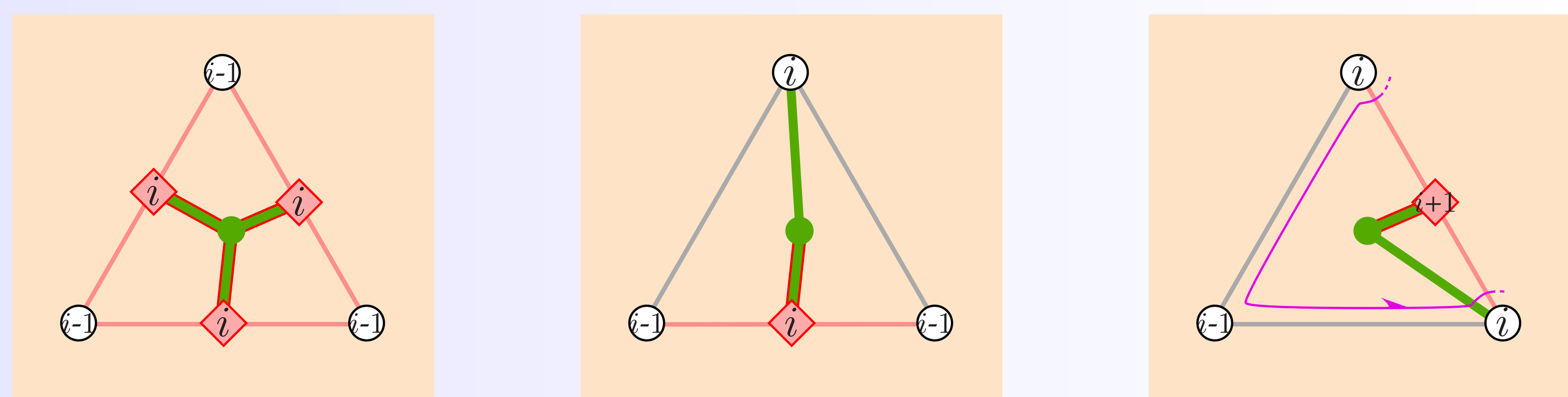
Reference

[CD15] G. Chapuy and M. Dołęga. A bijection for rooted maps on general surfaces. *Preprint, arXiv:1501.06942*, 2015.

Triangulations

The labels of the encoding maps satisfy local constraints. In general, they also satisfy a global "orientation" constraint. In the case of quadrangulations ([CD15]) and triangulations, the global constraint is automatically satisfied.

For triangulations, the encoding mobiles are unicellular labeled mobiles with green vertices of the following 3 possible types:



Prop. *The number of (rooted) triangulations with $2n$ faces of a surface $\mathcal{S}$ of type h is asymptotically equivalent to*

$$c_{\mathcal{S}}\, n^{5(h-1)/2}\big(12\sqrt{3}\big)^n .$$

Prop. *The generating function of triangulations counted with weight x per vertex is given by*

$$\frac{1}{2}(1-2\sigma)(1-\sigma+\sigma^2) - \frac{1}{2}\sqrt{1-6\sigma+6\sigma^2}$$

for the projective plane and by

$$3\sigma(1-\sigma)\big(1-6\sigma+6\sigma^2\big)^{-2}\Big(7-30\sigma+30\sigma^2-6(1-2\sigma)\sqrt{1-6\sigma+6\sigma^2}\Big)$$

for the Klein bottle, where σ is an algebraic function of x given by

$$x = \frac{1}{2}\sigma(1-\sigma)(1-2\sigma) \qquad \sigma(0) = 0.$$