A bijection for nonorientable maps

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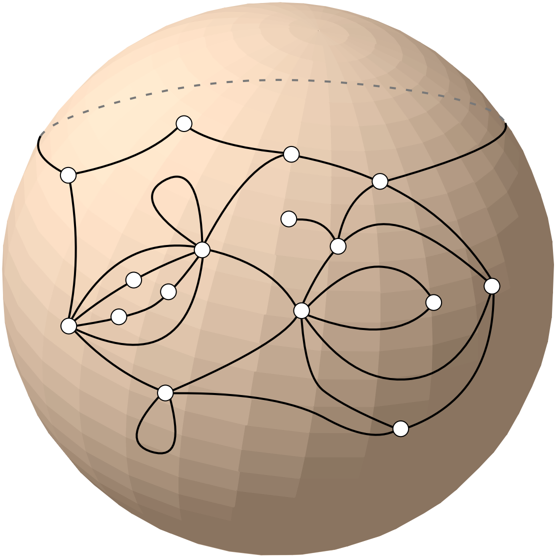
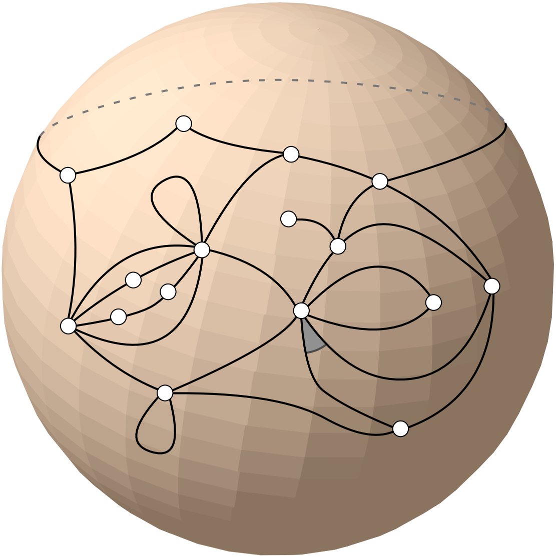
Plane maps



plane map: finite connected graph embedded in the sphere

faces: connected components of the complement

Plane maps

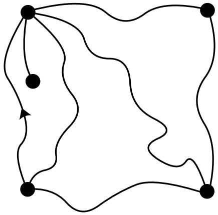


plane map: finite connected graph embedded in the sphere

faces: connected components of the complement

root: distinguished corner

Edge deformation



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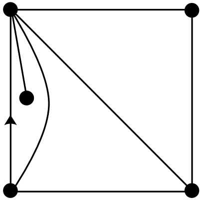

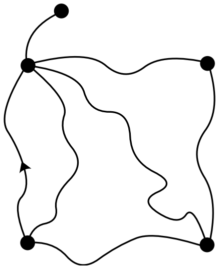

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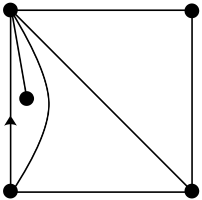

Scaling limit: the Brownian sphere

- We denote by $V(\mathfrak{m})$ the vertex-set of $\mathfrak{m}$ and $d_{\mathfrak{m}}$ the graph metric.

Theorem (Le Gall '11, Miermont '11)

Let $\mathfrak{q}_n$ be a uniform plane quadrangulation with n faces. The sequence $\left(V(\mathfrak{q}_n), (8n/9)^{-1/4}\, d_{\mathfrak{q}_n}\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space a.s. homeomorphic to the sphere, and called the Brownian sphere.

Definition (Convergence for the Gromov–Hausdorff topology)

A sequence $(\mathcal{X}_n)$ of compact metric spaces **converges in the sense of the Gromov–Hausdorff topology** toward a metric space $\mathcal{X}$ if there exist isometric embeddings $\varphi_n : \mathcal{X}_n \to \mathcal{Z}$ and $\varphi : \mathcal{X} \to \mathcal{Z}$ into a common metric space $\mathcal{Z}$ such that $\varphi_n(\mathcal{X}_n)$ converges toward $\varphi(\mathcal{X})$ in the sense of the Hausdorff topology.

Scaling limit: the Brownian sphere

- We denote by $V(\mathfrak{m})$ the vertex-set of $\mathfrak{m}$ and $d_{\mathfrak{m}}$ the graph metric.

Theorem (B. & Jacob & Miermont '13)

Let $\mathfrak{m}_n$ be a uniform plane map with n edges. The sequence $\left(V(\mathfrak{m}_n),(8n/9)^{-1/4}\,d_{\mathfrak{m}_n}\right)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space a.s. homeomorphic to the sphere, and called the Brownian sphere.

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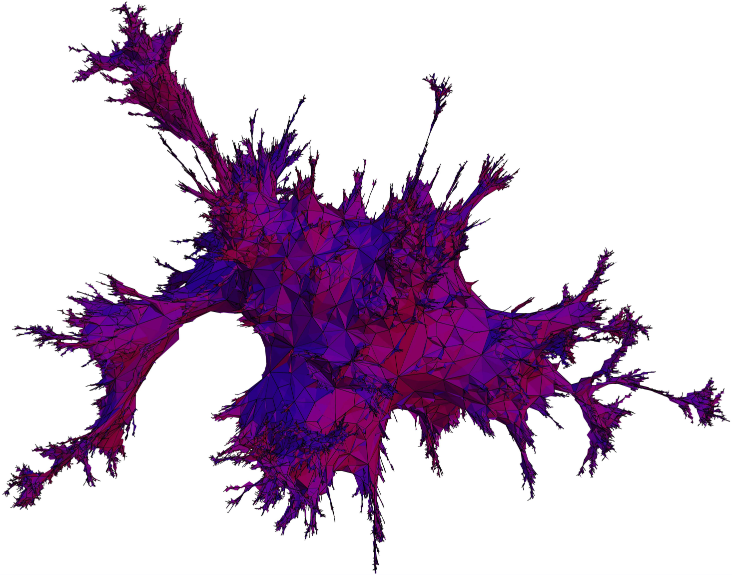
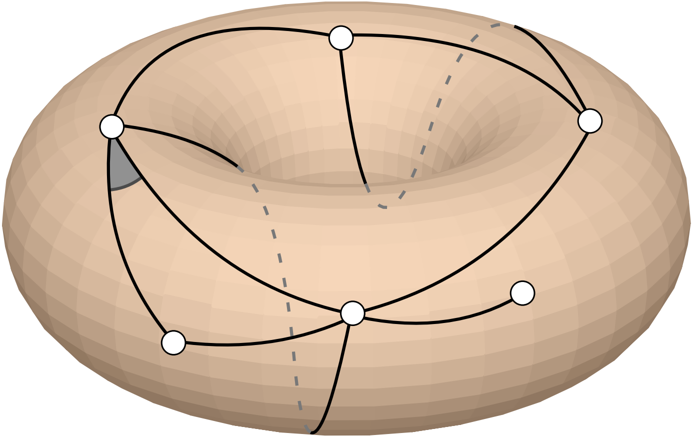
Uniform plane quadrangulation with 50 000 faces

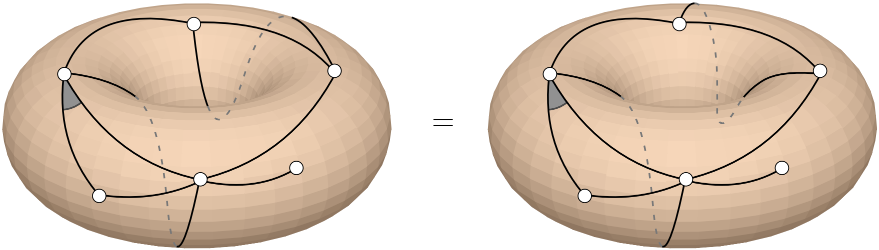


Genus g maps



genus g map: graph embedded in the surface of genus g, in such a way that the faces are homeomorphic to disks

Edge deformation



maps are defined up to direct homeomorphism of the underlying surface

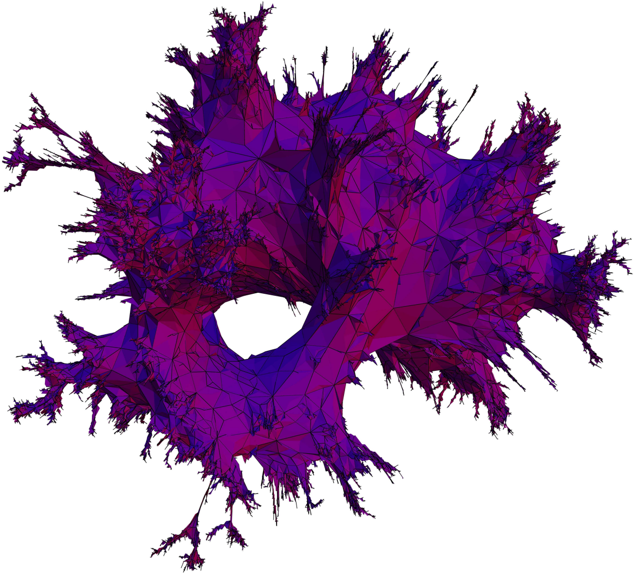
Scaling limit: the Brownian surface of genus g

- We denote by $V(\mathfrak{m})$ the vertex-set of $\mathfrak{m}$ and $d_{\mathfrak{m}}$ the graph metric.

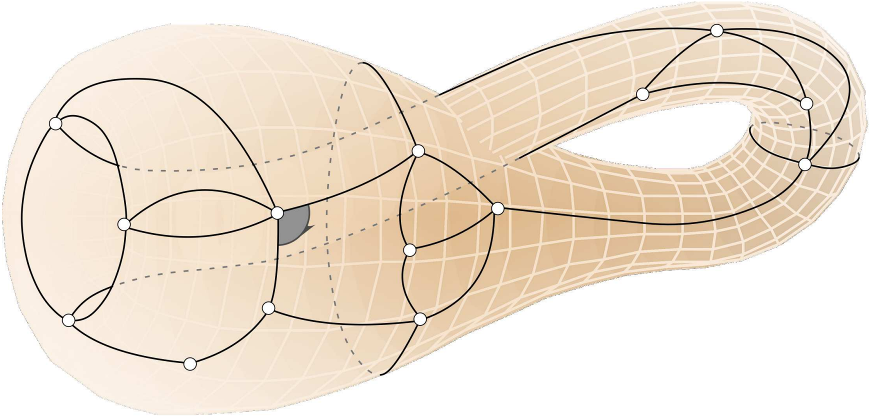
Theorem (B. & Miermont, in prep.)

Let $g \geq 1$ be fixed and $\mathfrak{q}_n$ be a uniform genus g quadrangulation with n faces. The sequence $\big(V(\mathfrak{q}_n), (8n/9)^{-1/4}\, d_{\mathfrak{q}_n}\big)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space a.s. homeomorphic to the surface of genus g, and called the Brownian surface of genus g.

Uniform genus 1 quadrangulation with 50 000 faces



Nonorientable maps



root: distinguished corner given with a local orientation

maps are defined up to homeomorphism of the underlying surface

Scaling limit: Brownian nonorientable surfaces

- We denote by $V(\mathfrak{m})$ the vertex-set of $\mathfrak{m}$ and $d_{\mathfrak{m}}$ the graph metric.

Theorem (Chapuy & Dołęga '17)

Let $\mathcal{S}$ be a fixed nonorientable surface and $\mathfrak{q}_n$ be a uniform quadrangulation of $\mathcal{S}$ with n faces. Up to extraction, the sequence $\big(V(\mathfrak{q}_n), (8n/9)^{-1/4}\, d_{\mathfrak{q}_n}\big)_{n\geq 1}$ converges weakly in the sense of the Gromov–Hausdorff topology toward a random compact metric space.

Enumeration results in the case of triangulations

Proposition

The number of (rooted) triangulations of $\mathcal{S}$ with $2n$ faces (and thus $3n$ edges and $n+2-2h$ vertices) is asymptotically equivalent to

$$c_{\mathcal{S}}\, n^{5(h-1)/2} \big(12\sqrt{3}\big)^n\,,$$

where h is the type of $\mathcal{S}$ and $c_{\mathcal{S}}$ is a constant that depends on $\mathcal{S}$.

h	$\mathcal{S}$	$c_{\mathcal{S}}$
0	sphere	$\frac{\sqrt{6}}{\sqrt{\pi}}$
1	torus	$\frac{1}{8}$

h	$\mathcal{S}$	$c_{\mathcal{S}}$
$\frac{1}{2}$	projective plane	$\frac{2^{-3/4}\,3^{5/4}}{\Gamma(3/4)}$
1	Klein bottle	$\frac{3}{2}$

Enumeration results in the case of triangulations

Proposition

The gfun of triangulations counted with weight x per vertex is

sphere: $$\frac{1}{2}\sigma^3(1-\sigma)(1-4\sigma+2\sigma^2)$$

proj. plane: $$\frac{1}{2}(1-2\sigma)(1-\sigma+\sigma^2)-\frac{1}{2}\sqrt{1-6\sigma+6\sigma^2}$$

torus: $$\frac{1}{2}\sigma(1-\sigma)\big(1-6\sigma+6\sigma^2\big)^{-2}$$

Klein bootle:
$$3\sigma(1-\sigma)\big(1-6\sigma+6\sigma^2\big)^{-2}\Big(7-30\sigma+30\sigma^2-6(1-2\sigma)\sqrt{1-6\sigma+6\sigma^2}\Big)$$

where σ *is given by* $x=\frac{1}{2}\sigma(1-\sigma)(1-2\sigma)$ *and* $\sigma(0)=0$.

A history of bijections

Encoding of pointed maps

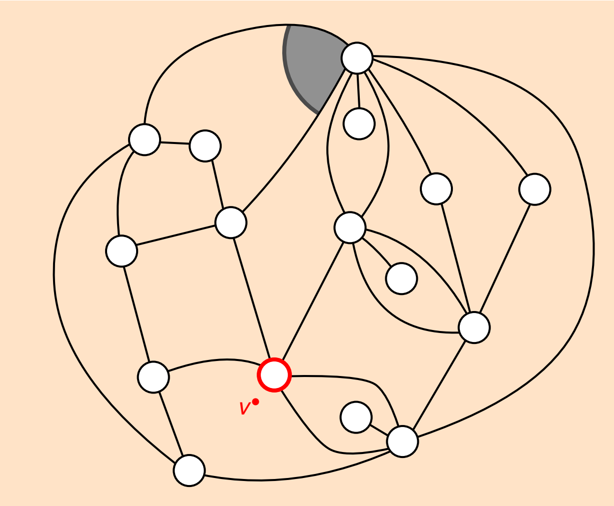
	sphere	orientable	nonorientable
bip. quad.	[CV'81] – [S'98]	[CMS'09]	[CD'17]
general maps	[BDG'04]	[BDG'04]+[CMS'09]	[B'16]

[CV'81]: Cori–Vauquelin
[S'98]: Schaeffer
[BDG'04]: Bouttier–Di Francesco–Guitter
[CMS'09]: Chapuy–Marcus–Schaeffer
[CD'17]: Chapuy–Dołęga
[B'16]: this talk

Similar bijections

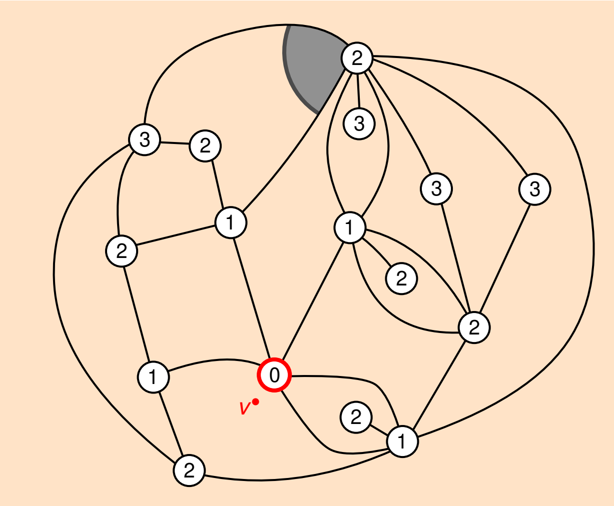
Multi-pointed quadranguations: [Miermont '09]
CVS with rules inverted: [Ambjørn–Budd '13]

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



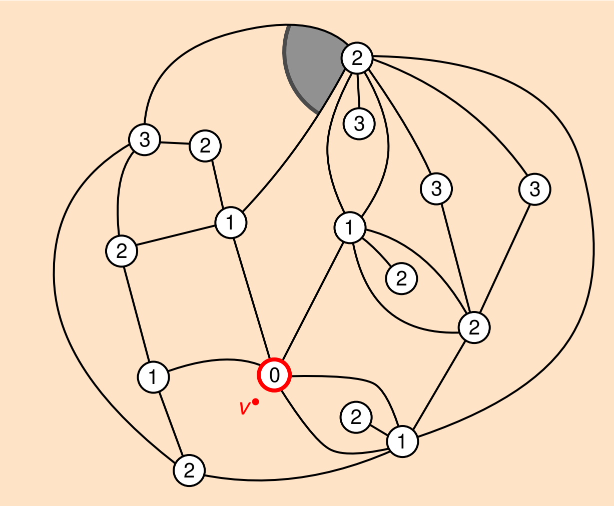
- Start with a pointed quadrangulation.

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer

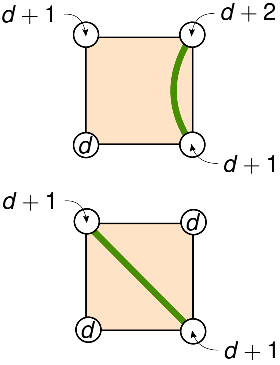


- Start with a pointed quadrangulation.
- Label the vertices with their distance to $v^\bullet$.

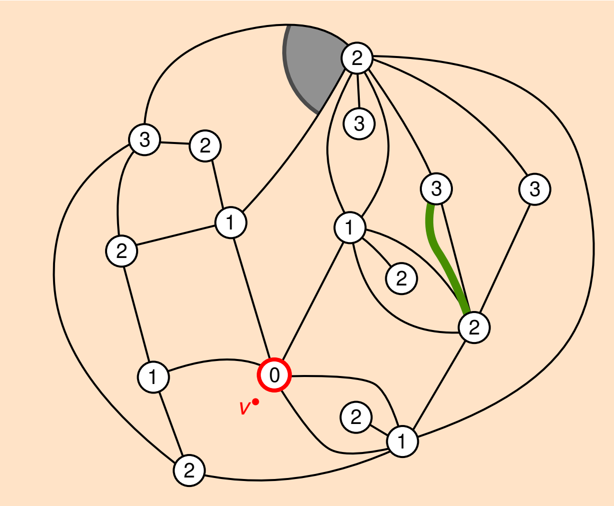
Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



- Apply the rule:



Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



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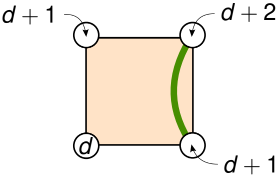
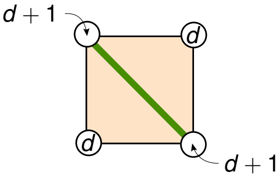
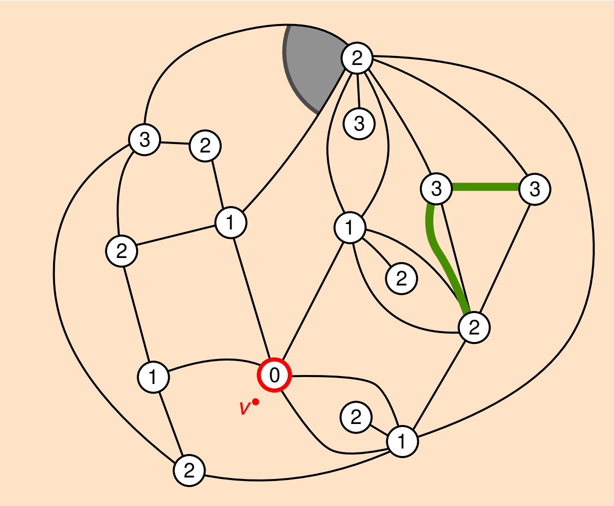




Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



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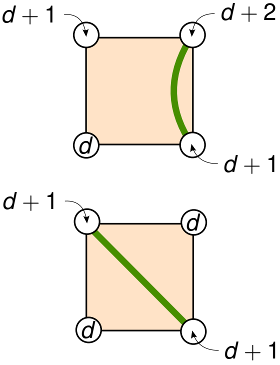

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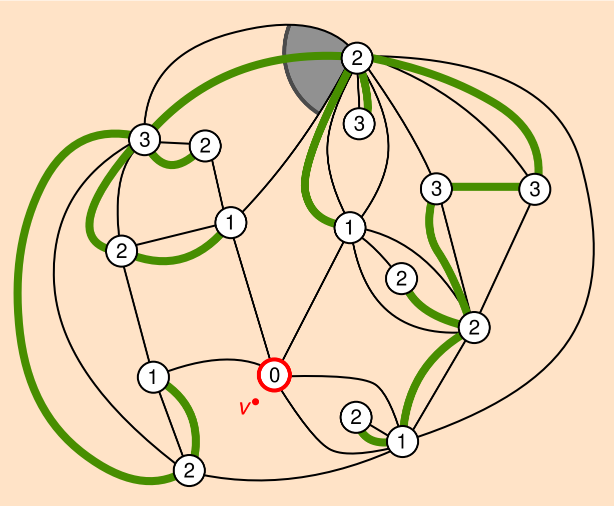

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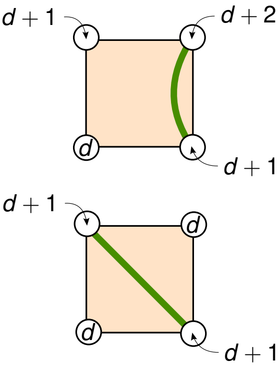
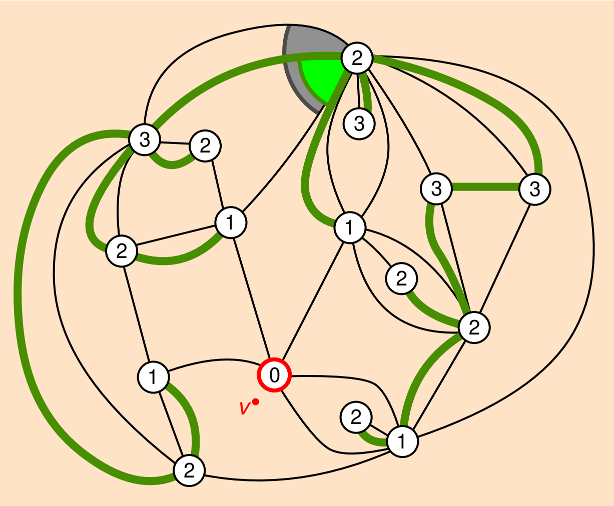
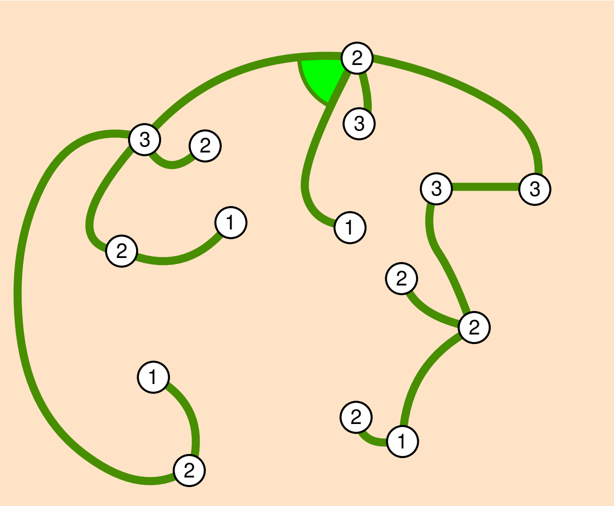


Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



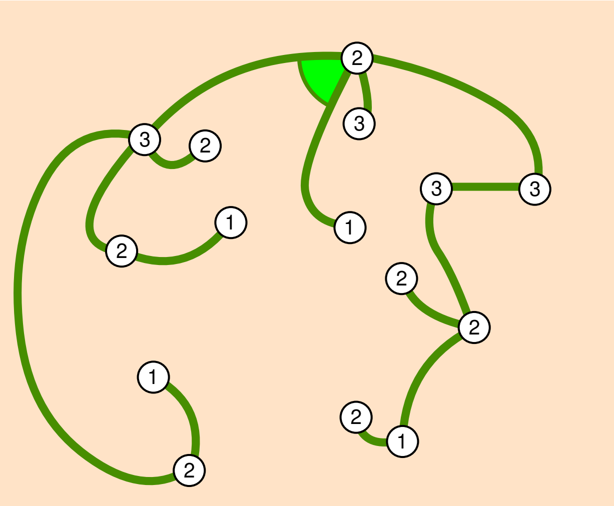
- Start with a pointed quadrangulation.
- Label the vertices with their distance to $v^{\bullet}$.
- Apply the rule.
- Root.

Cori–Vauquelin–Schaeffer / Chapuy–Marcus–Schaeffer



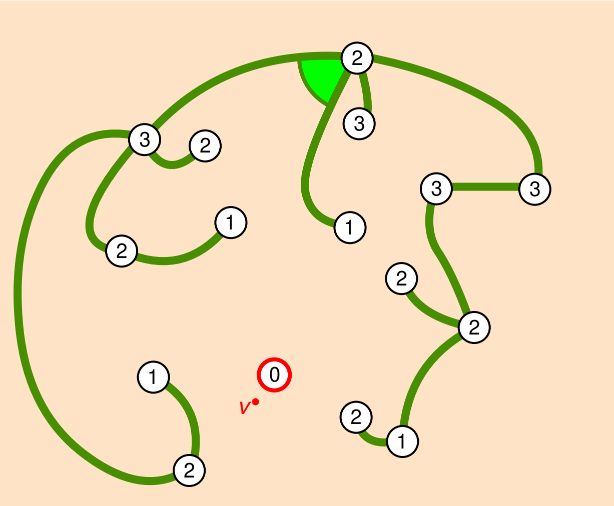
- Start with a pointed quadrangulation.
- Label the vertices with their distance to $v^\bullet$.
- Apply the rule.
- Root.
- Remove the initial edges and $v^\bullet$.

Inverse construction



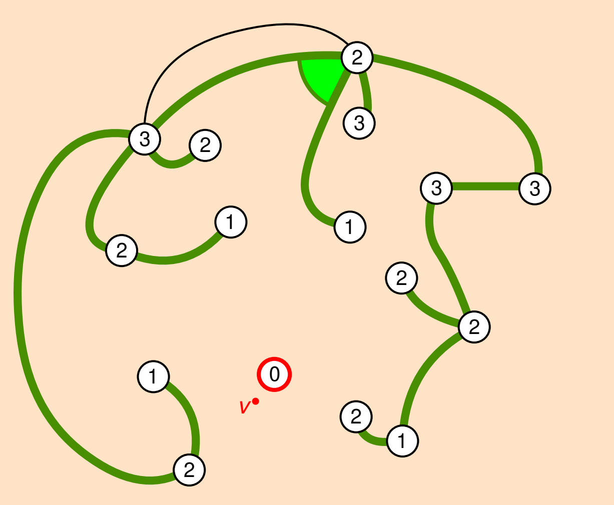
- Take a well-labeled unicellular map.

Inverse construction



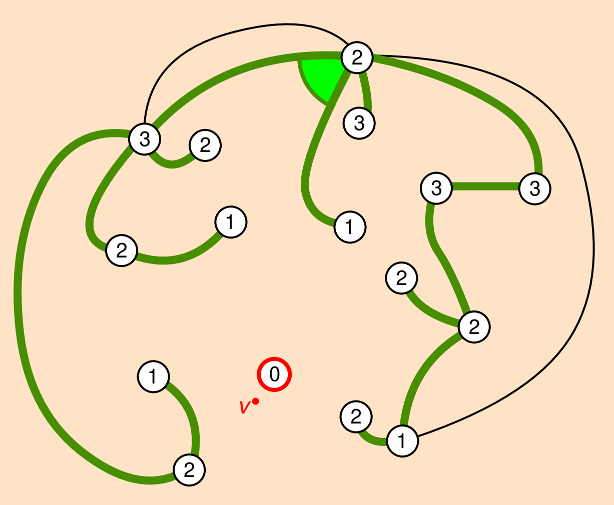
- Take a well-labeled unicellular map.
- Add a vertex $v^{\bullet}$ inside the unique face.

Inverse construction



- Take a well-labeled unicellular map.
- Add a vertex $v^\bullet$ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.

Inverse construction



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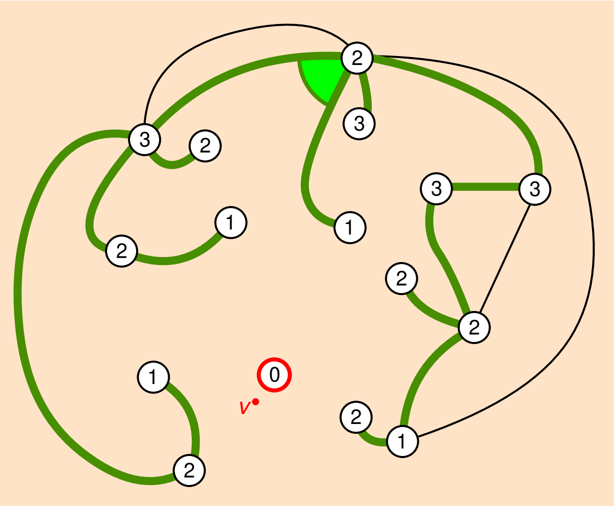

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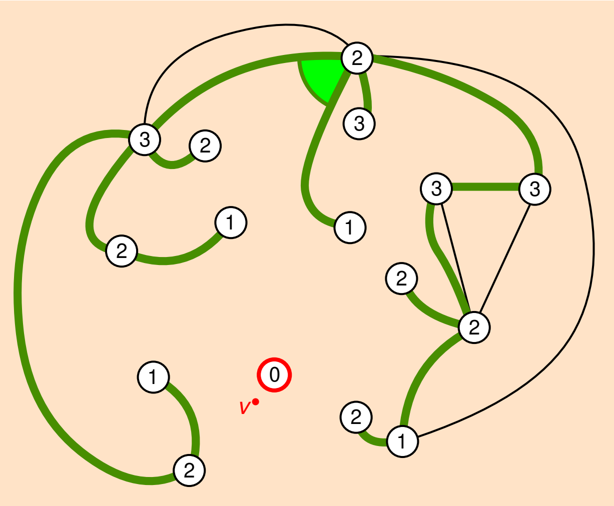

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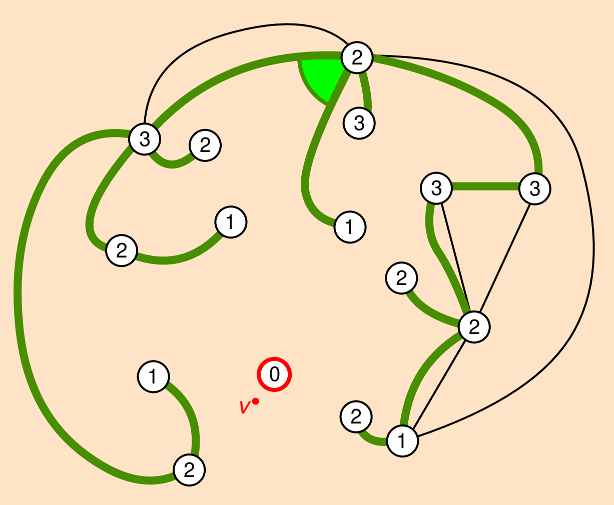

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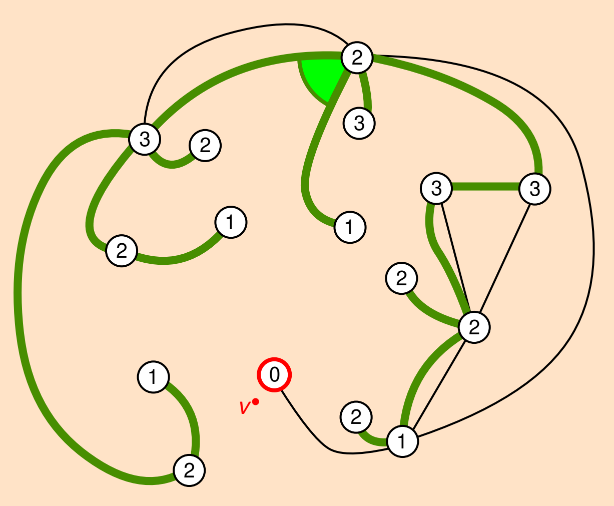

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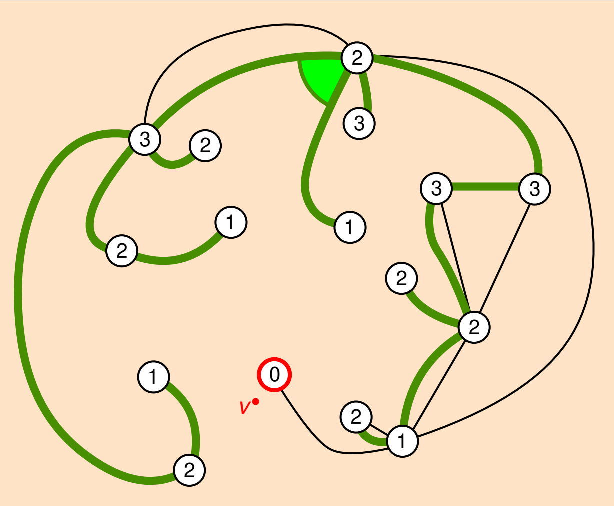

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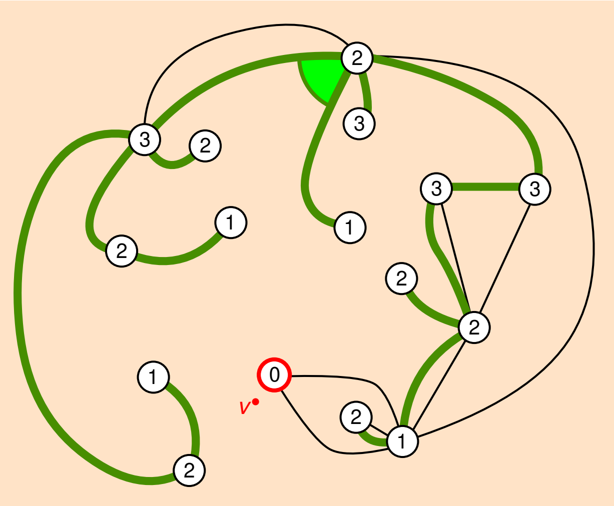

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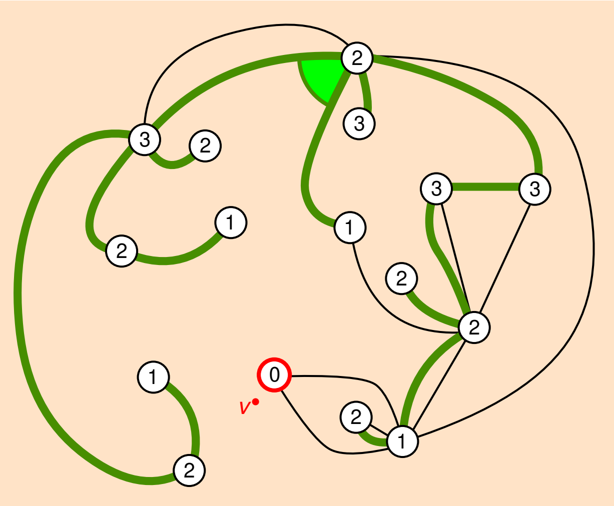

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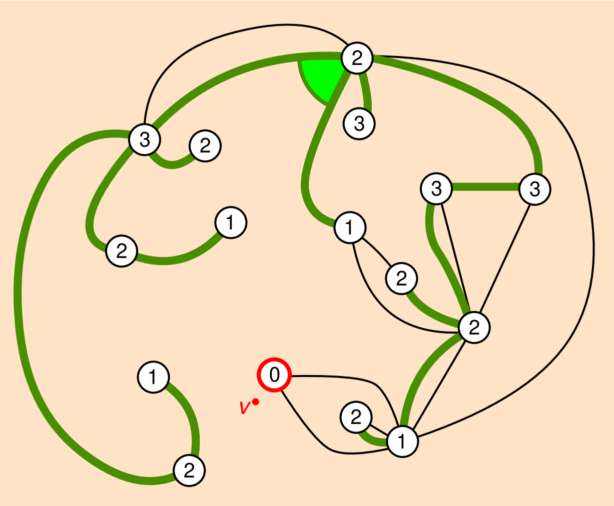

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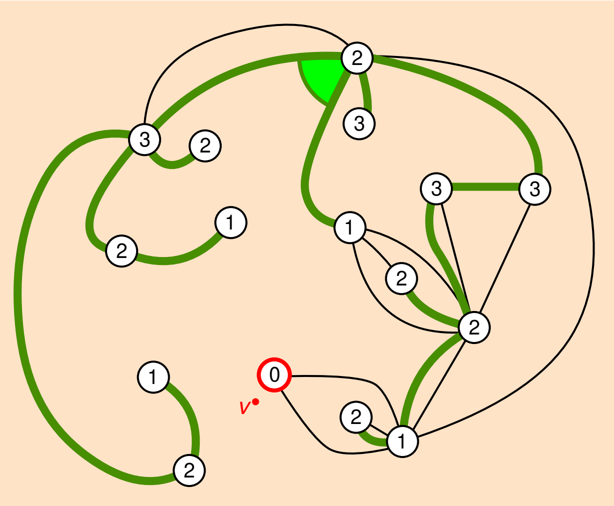

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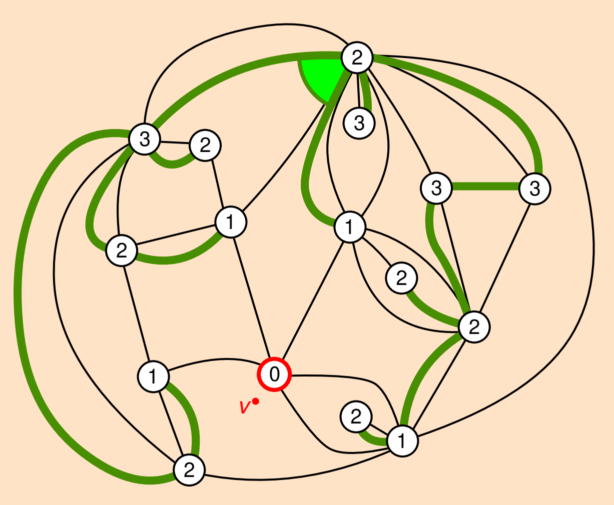

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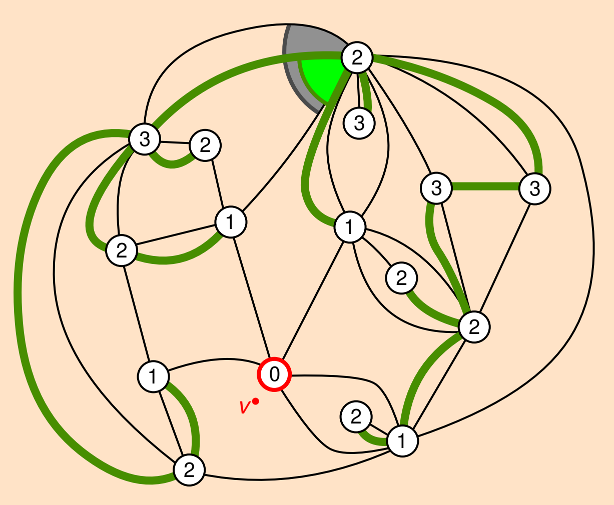
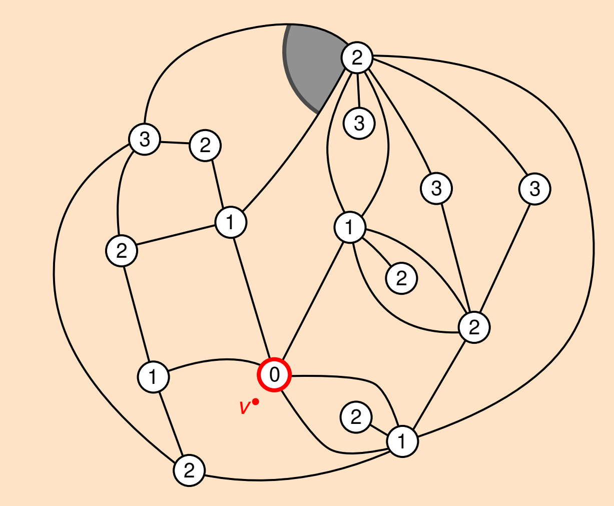


- Take a well-labeled unicellular map.
- Add a vertex $v^\bullet$ inside the unique face.
- Link every corner to the first subsequent corner having a strictly smaller label.
- Root and remove the initial edges.

Inverse construction



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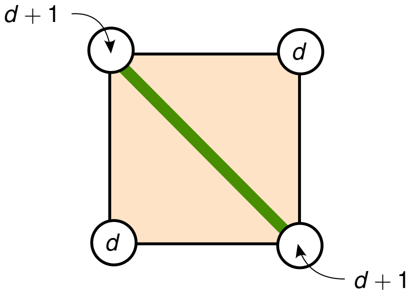
What could go wrong with nonorientable maps?

From quadrangulations to unicellular maps



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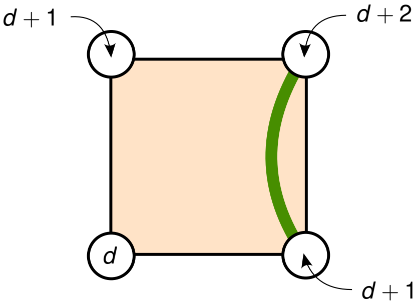
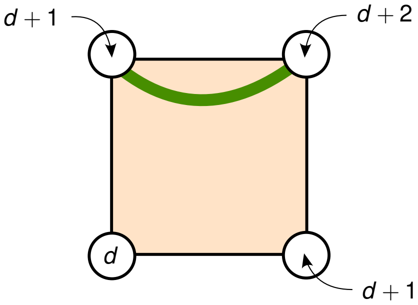


or



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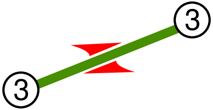

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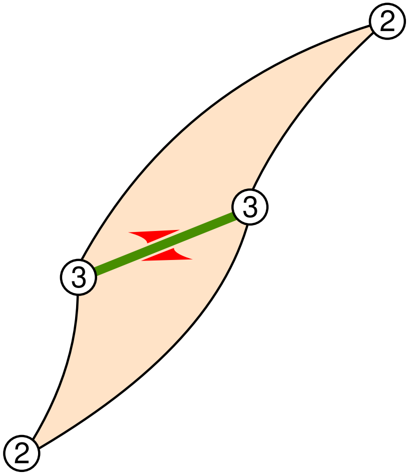

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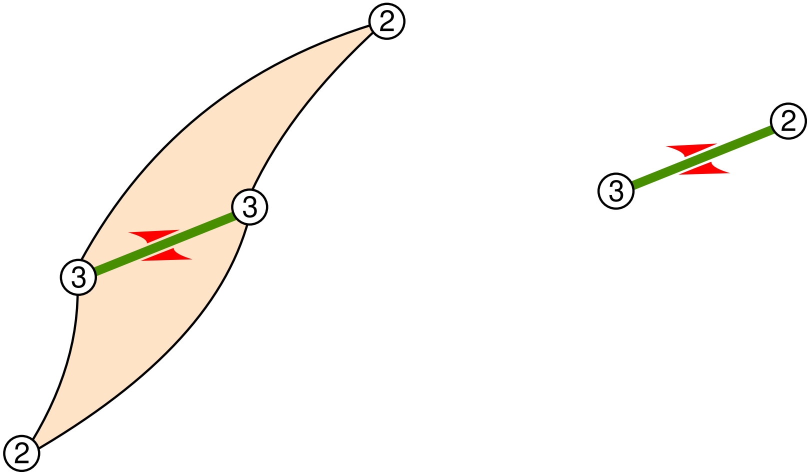

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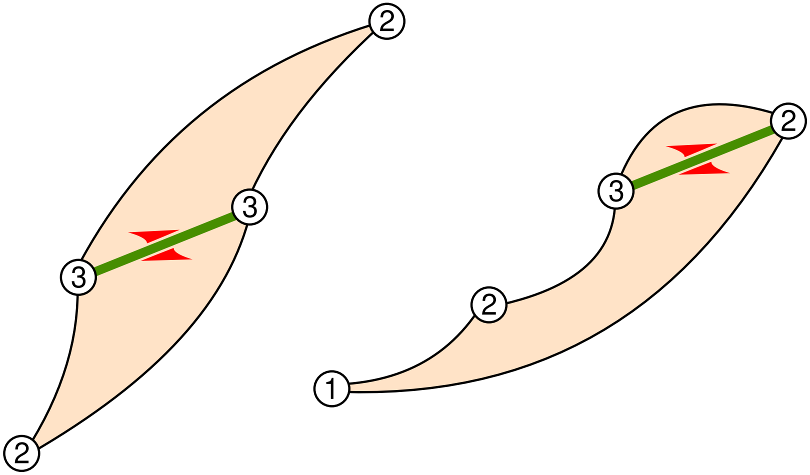

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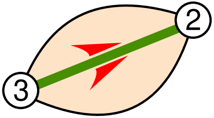
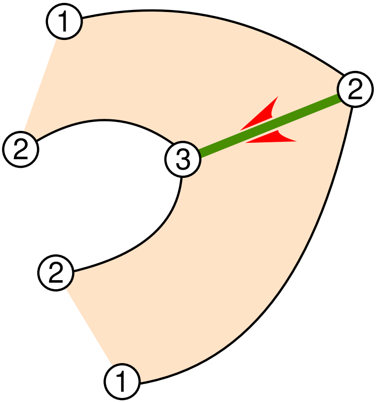
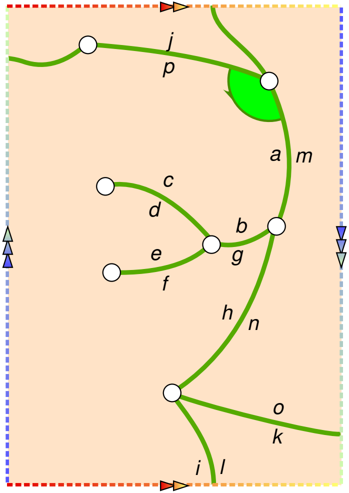
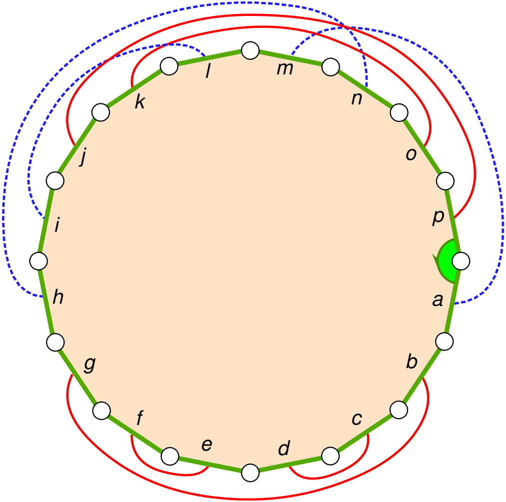
From unicellular maps to quadrangulations

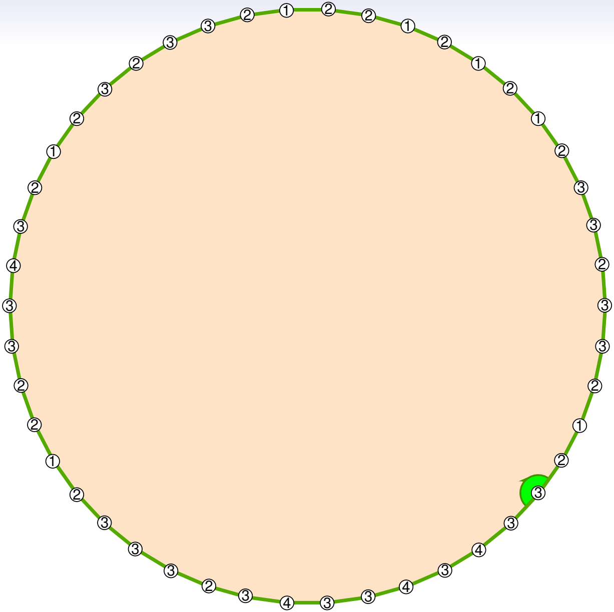


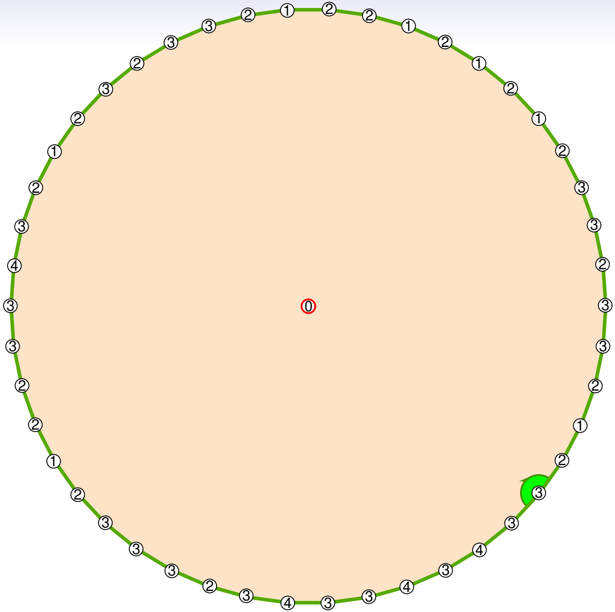


Unicellular maps seen as polygons with paired sides





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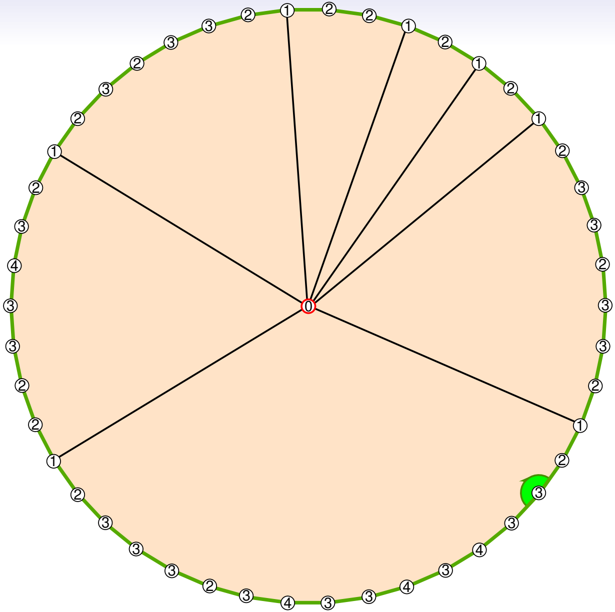

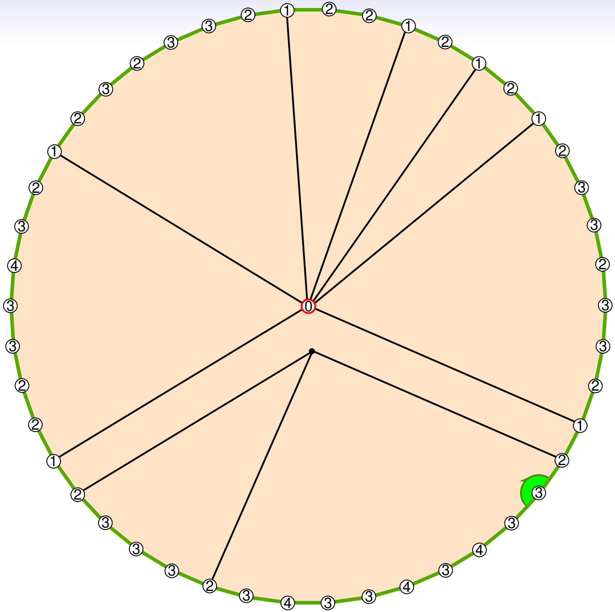
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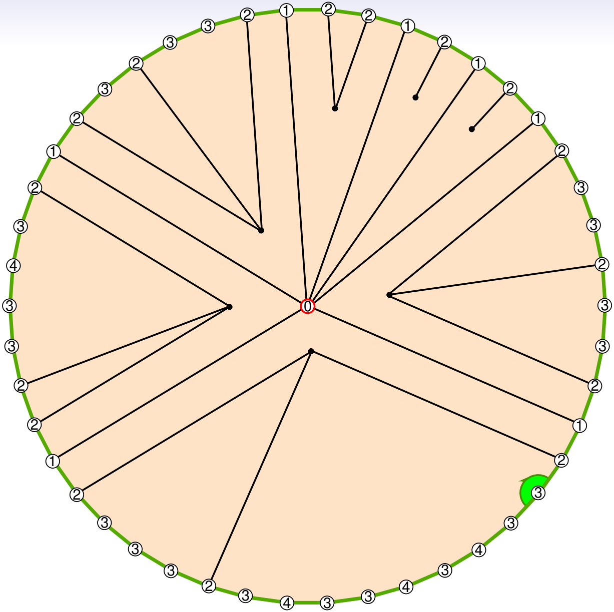
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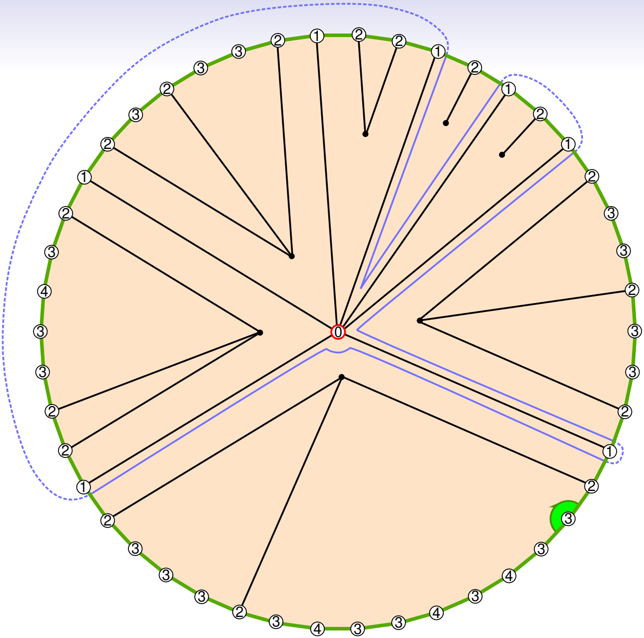
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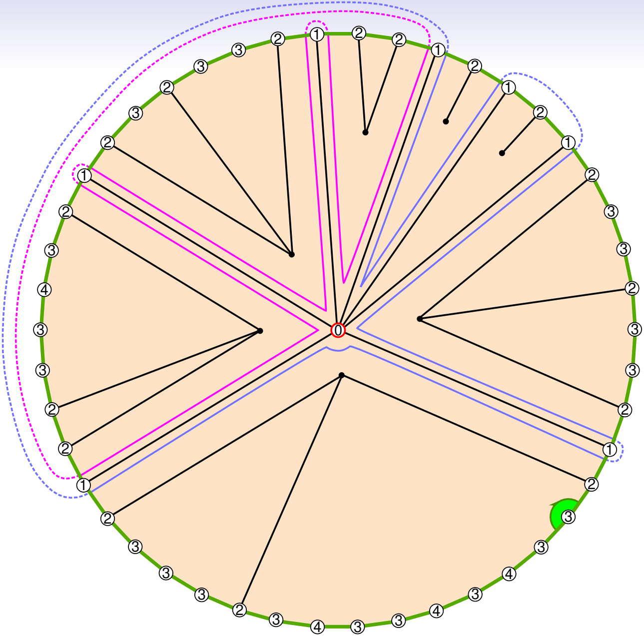

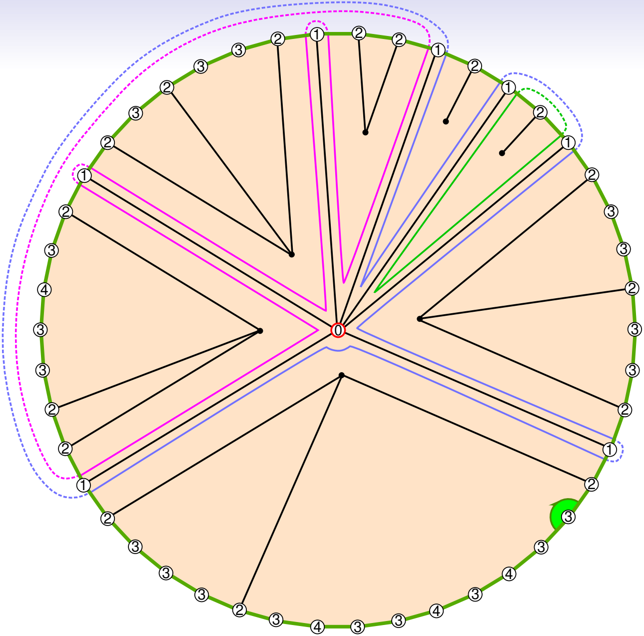
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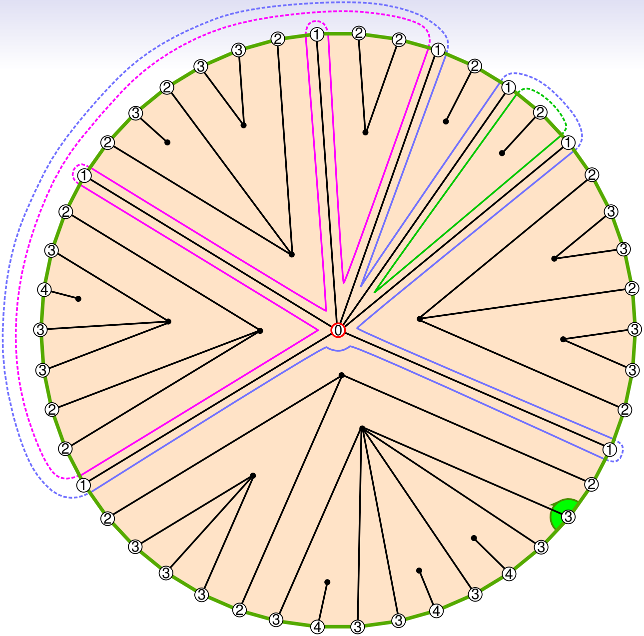

Labeled unicellular mobiles

Definition (Labeled unicellular mobile)

A labeled unicellular mobile is a pair $(\mathfrak{u}, \mathfrak{l})$ such that

- $\mathfrak{u}$ is a one-face map with vertex set $V_\bullet(\mathfrak{u}) \sqcup V_\circ(\mathfrak{u})$ and only edges linking vertices from $V_\bullet(\mathfrak{u})$ to vertices from $V_\circ(\mathfrak{u})$;
- $\mathfrak{l} : V_\circ(\mathfrak{u}) \to \mathbb{N}$ is a function with minimum 1;
- the root vertex belongs to $V_\circ(\mathfrak{u})$.

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Definition (Well-labeled unicellular mobile in the orientable case)

The mobile $(\mathfrak{u}, \mathfrak{l})$ is well labeled if, for every white corner $\mathbb{c}$, the label of the first subsequent corner is $\geq \mathfrak{l}(\mathbb{c}) - 1$.

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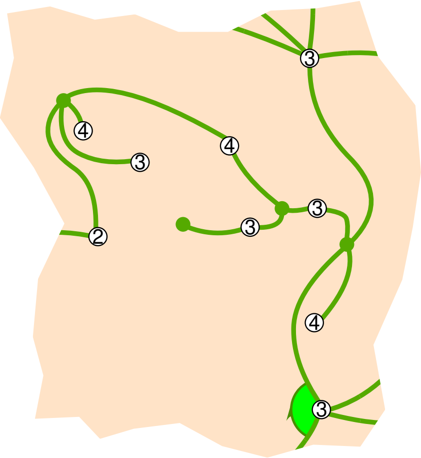
Coherent orientation of the corners



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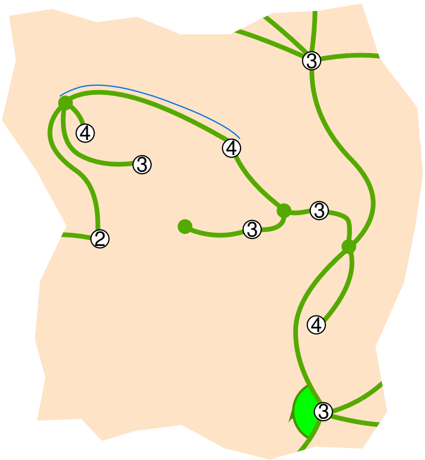
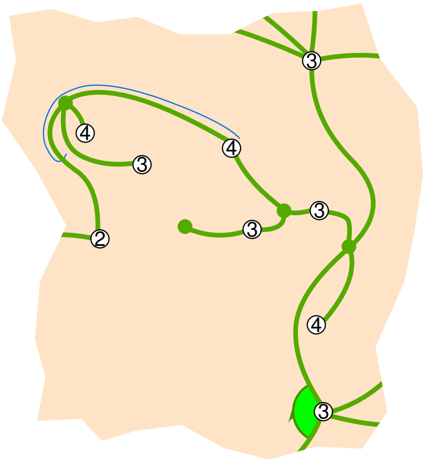


- Start from a white corner $\mathbb{C}$, arbitrarily oriented, and move to the next green corner.

Coherent orientation of the corners



- Start from a white corner $\mathbb{c}$, arbitrarily oriented, and move to the next green corner.
- Look at the next white corner:
  - if its label is $< \mathfrak{l}(\mathbb{c})$, cross the green edge;

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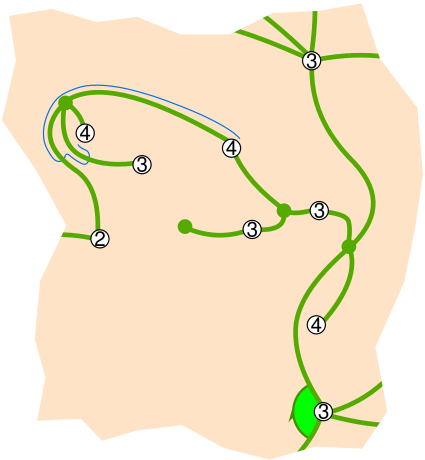

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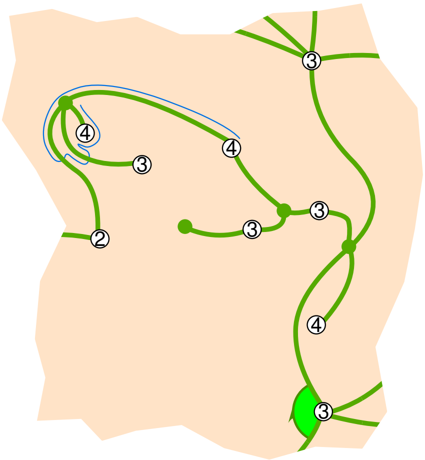

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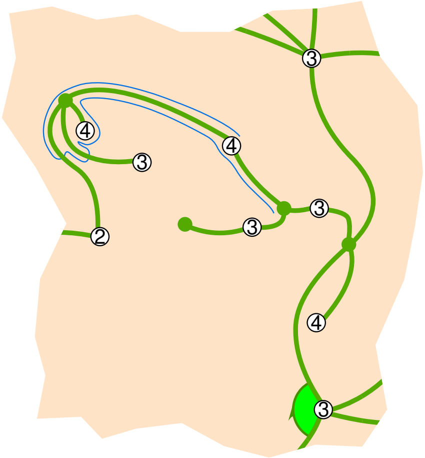

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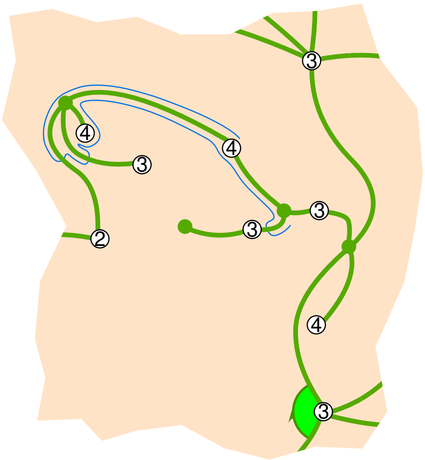

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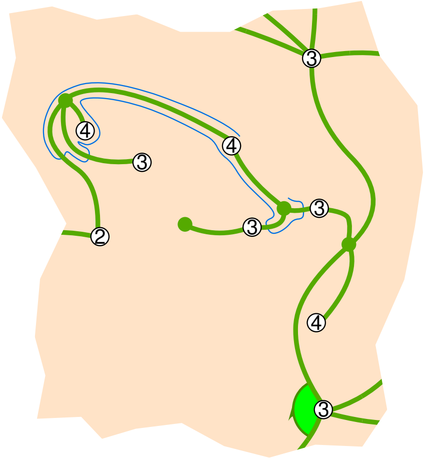

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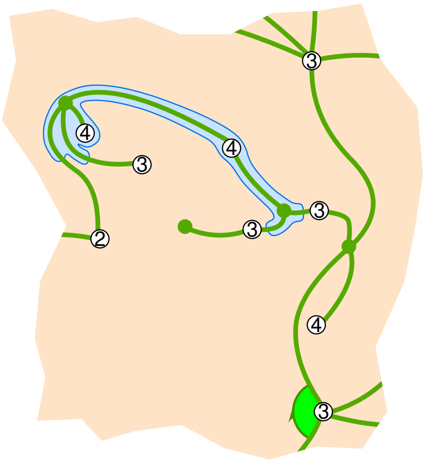

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- Get a corner cycle.

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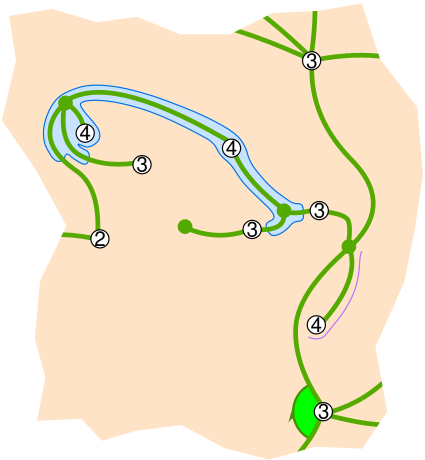

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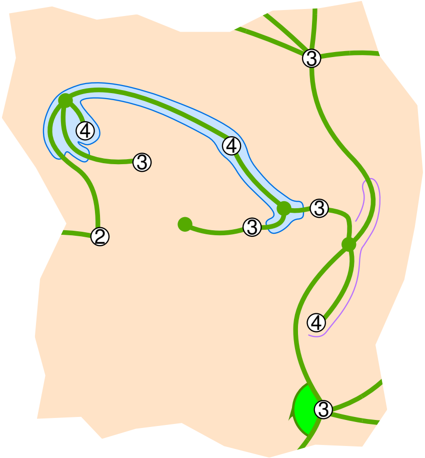

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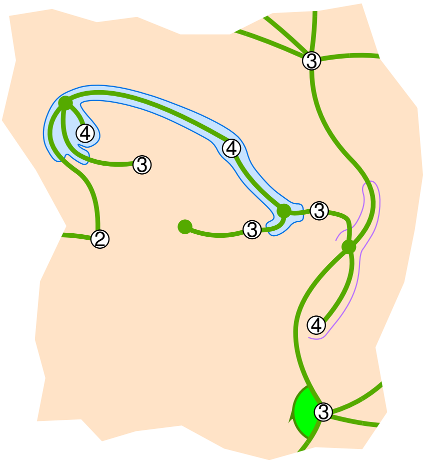

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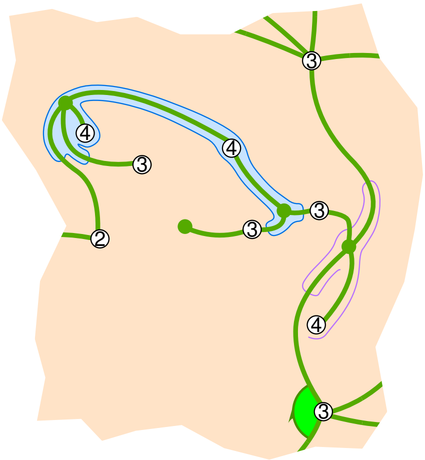

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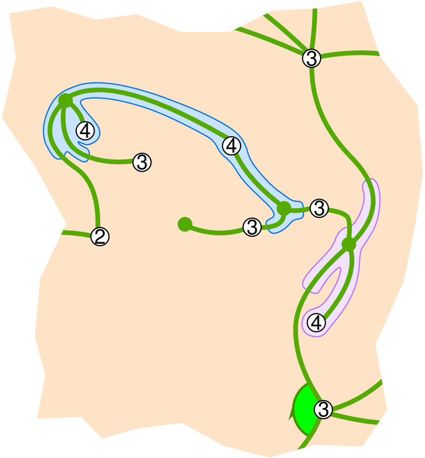

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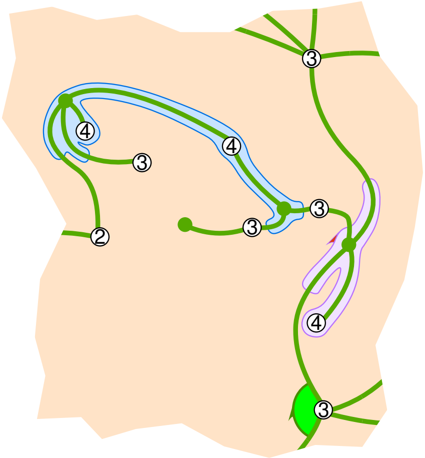

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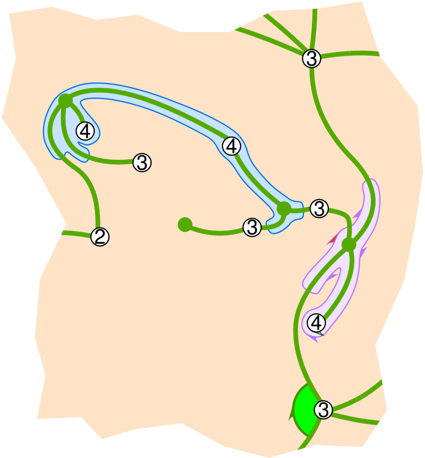
- Get a corner cycle.
- Consider the first subsequent corner of the root in the cycle.

Coherent orientation of the corners



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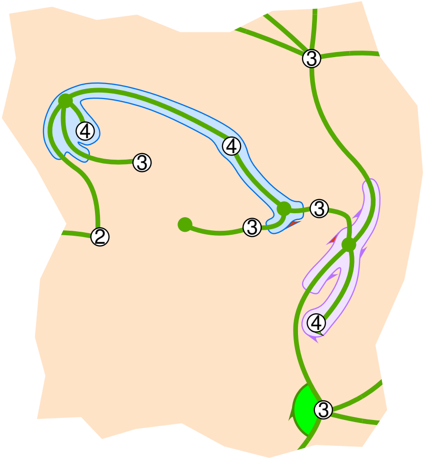
- Consider the first subsequent corner of the root in the cycle.
- Orient the cycle and $\mathbb{c}$.

Coherent orientation of the corners



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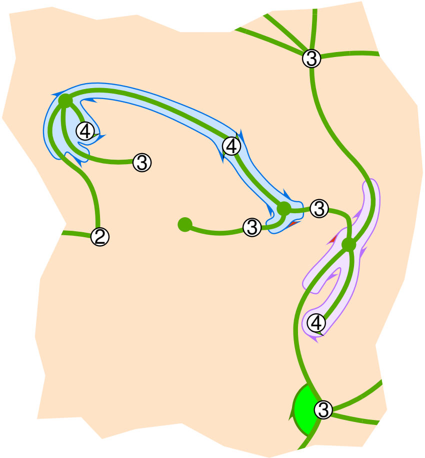

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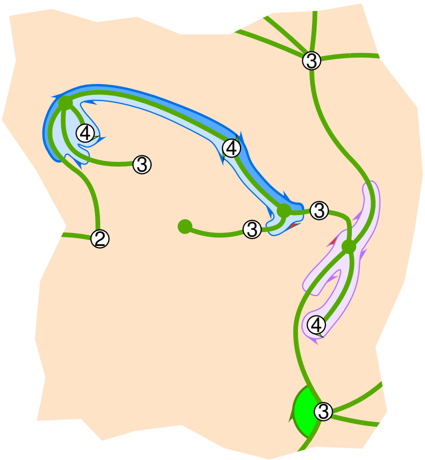

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- Orient the cycle and $\mathbb{c}$.
- A corner run.

Well-labeled unicellular mobiles

Definition (Successor)

Let $\mathbb{c}$ be a white corner with label $\mathfrak{l}(\mathbb{c}) \geq 2$. In the order given by the coherent orientation of $\mathbb{c}$, the first subsequent corner with label strictly smaller than $\mathfrak{l}(\mathbb{c})$ is called the successor of $\mathbb{c}$; it is denoted by $\mathsf{succ}(\mathbb{c})$.

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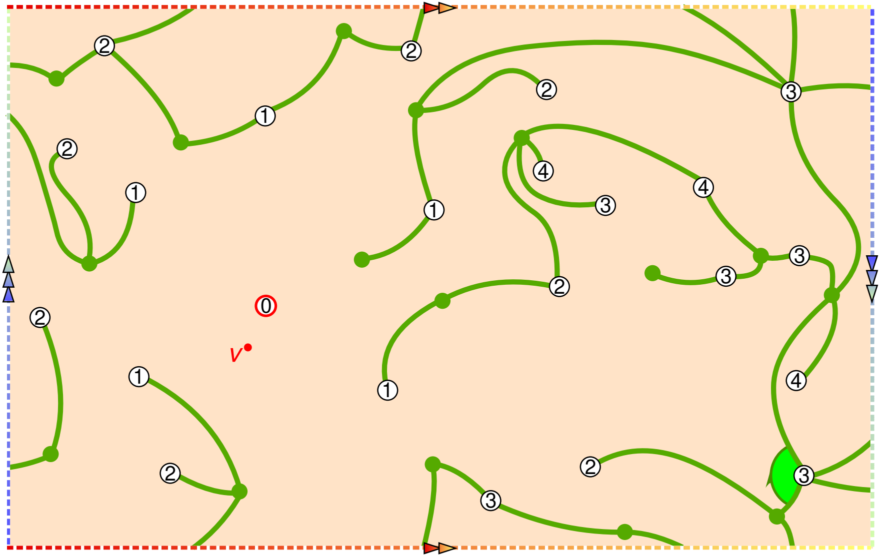
$$\mathfrak{l}(\mathsf{succ}(\mathbb{c})) = \mathfrak{l}(\mathbb{c}) - 1.$$

From unicellular mobiles to pointed bipartite maps



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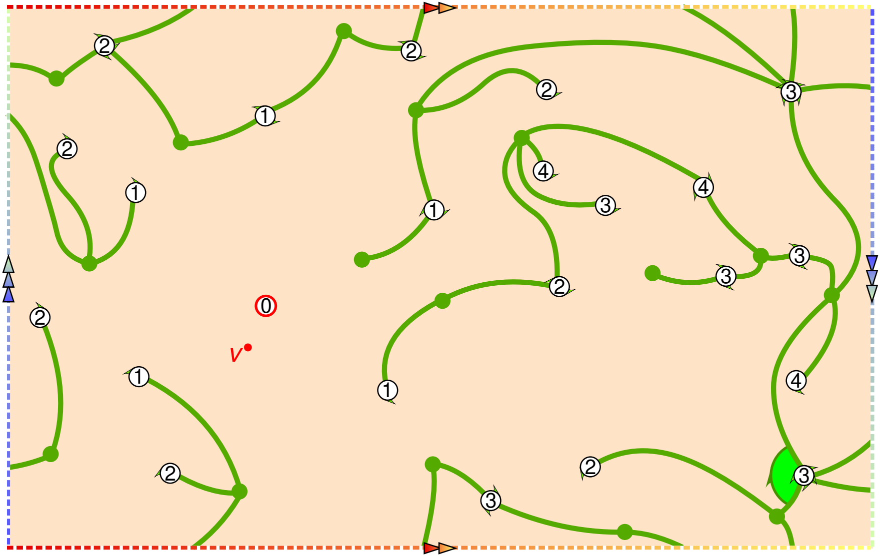

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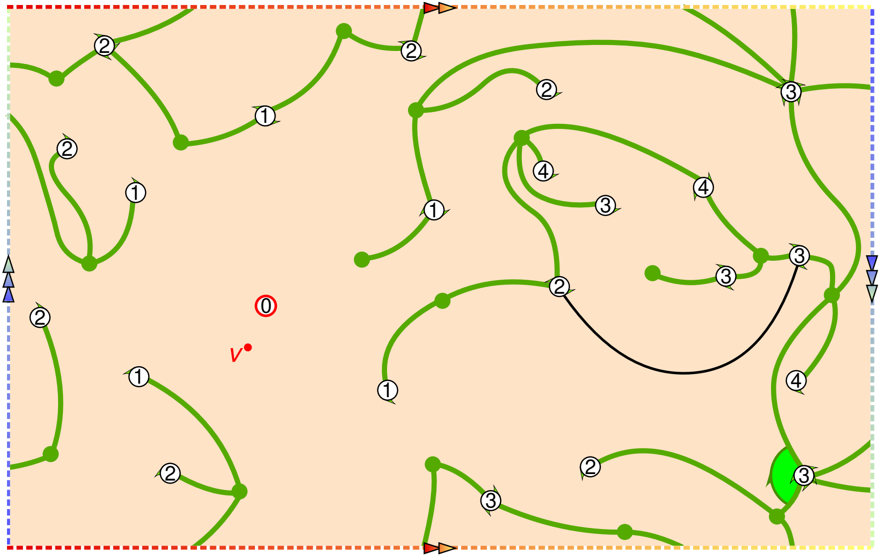

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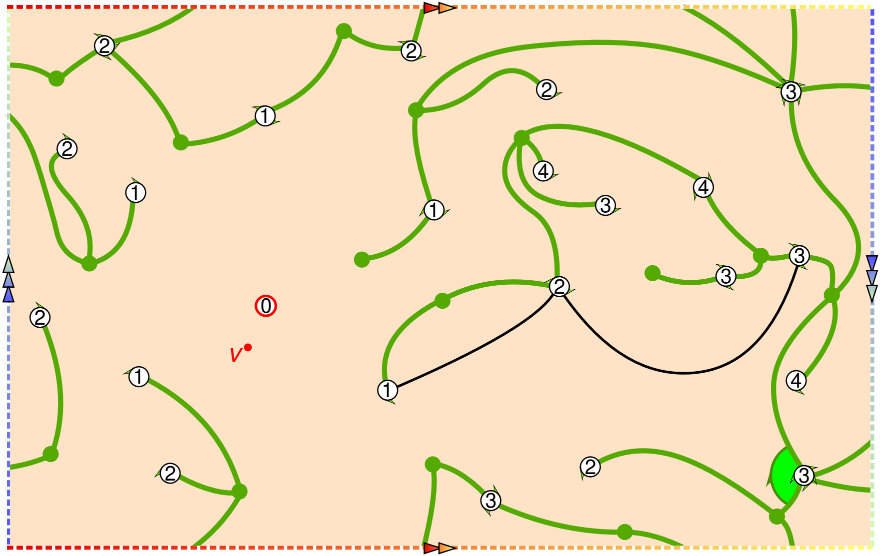

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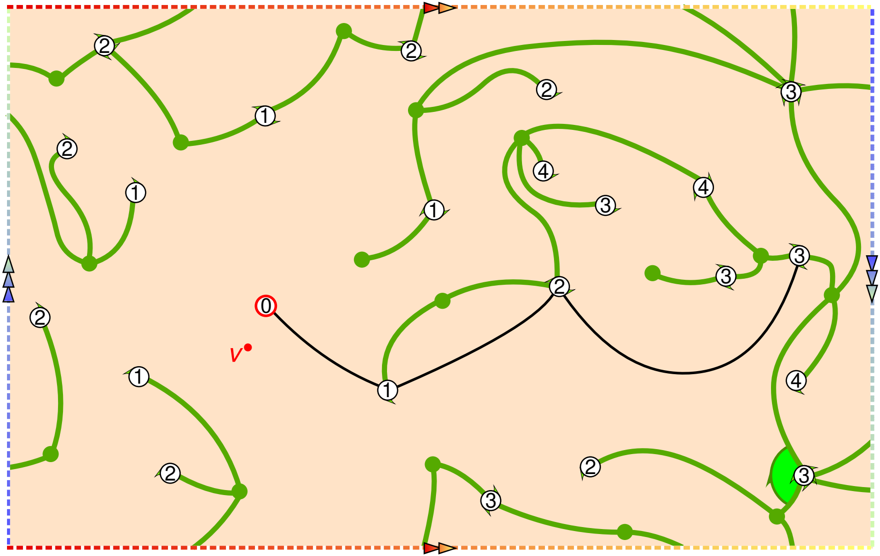

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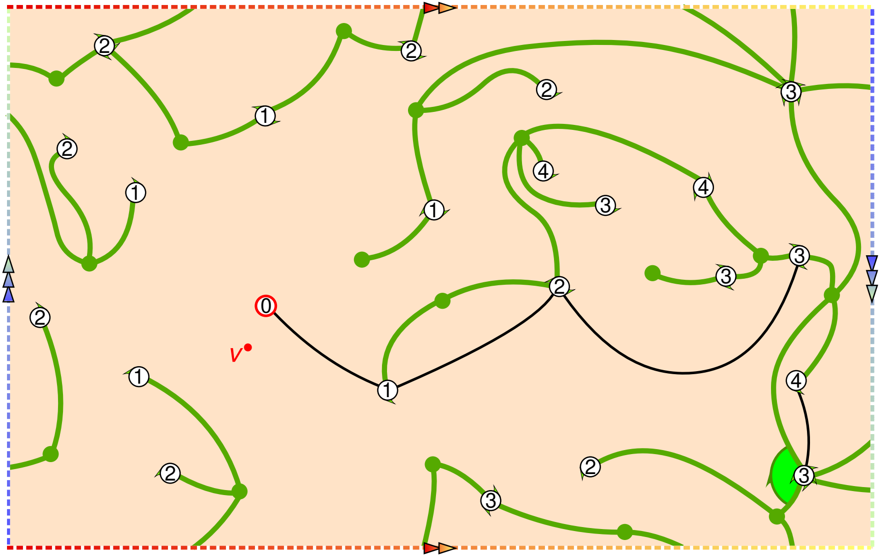

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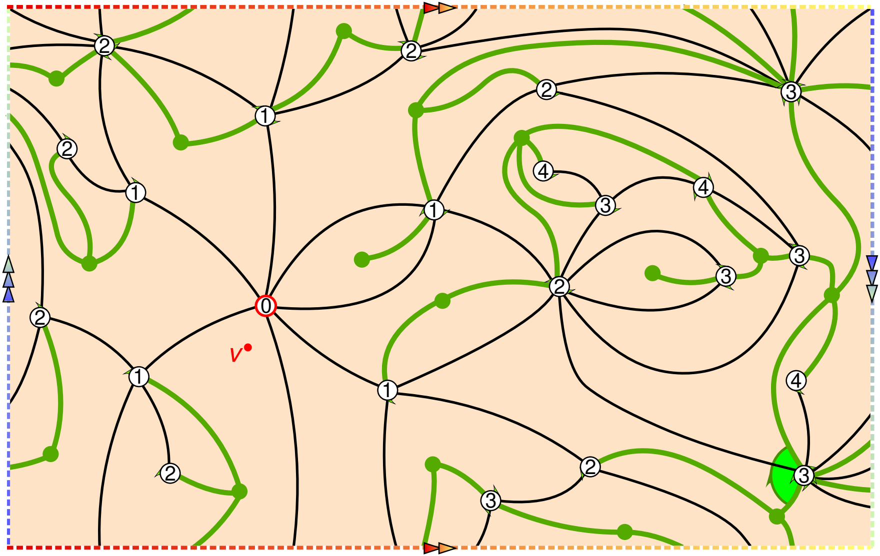

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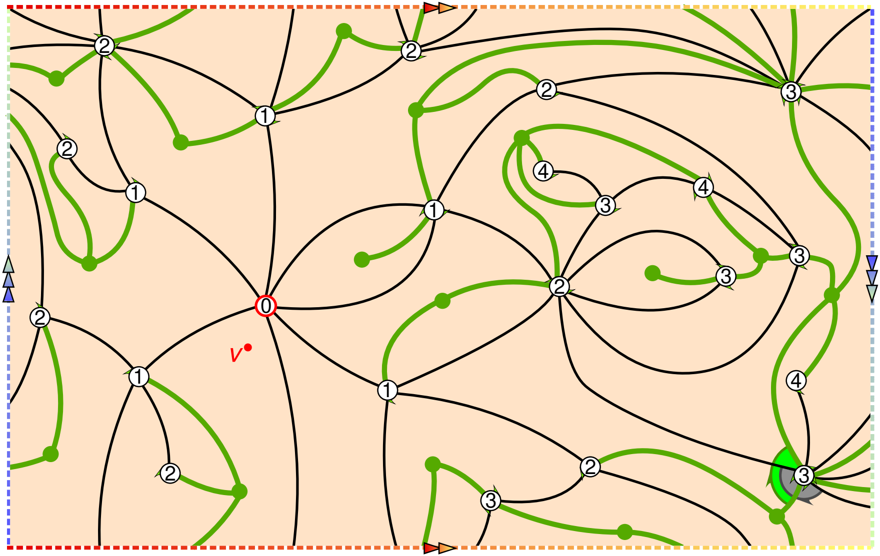

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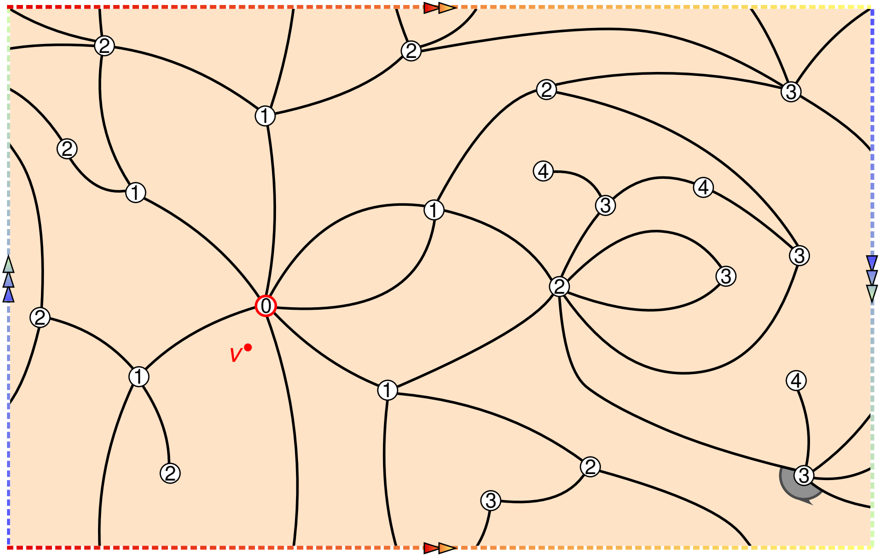
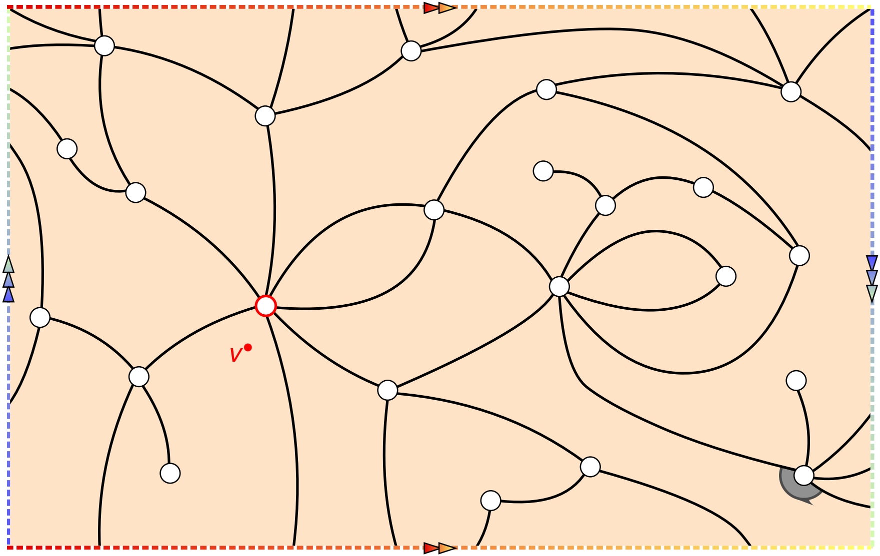


Level loops



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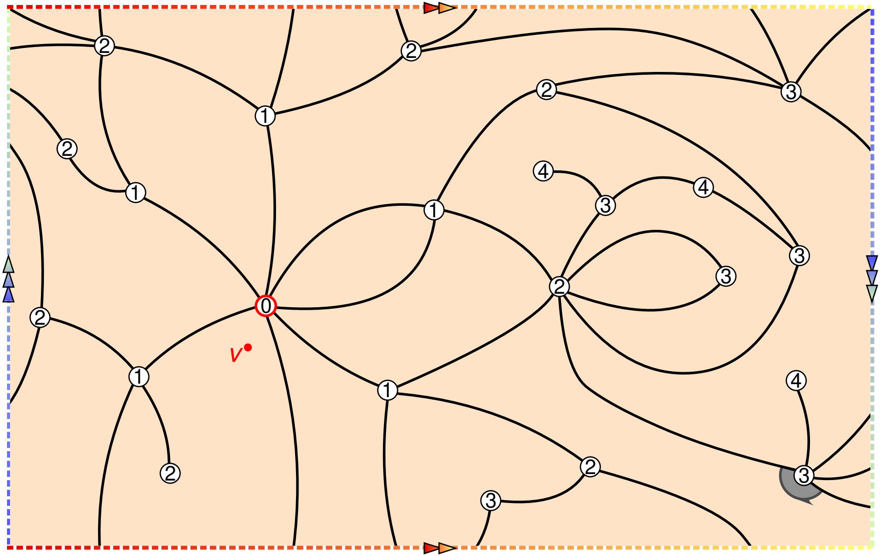

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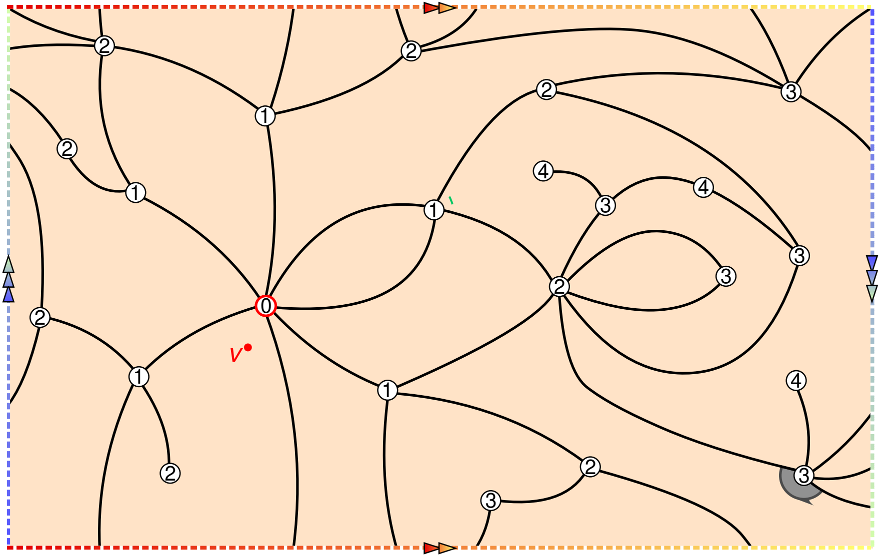
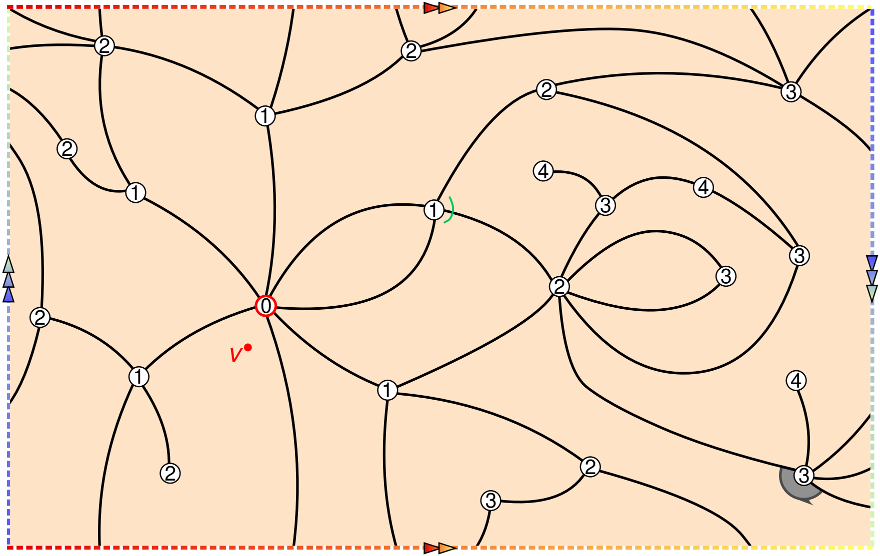
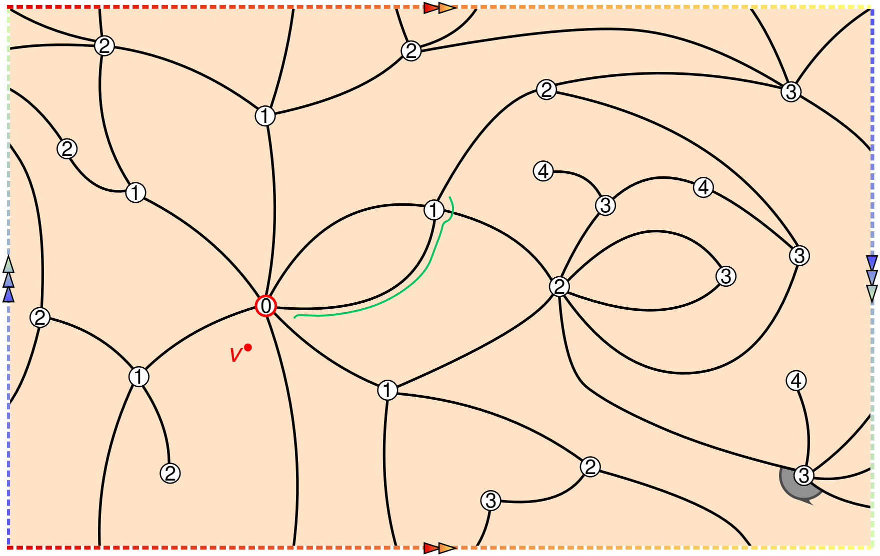


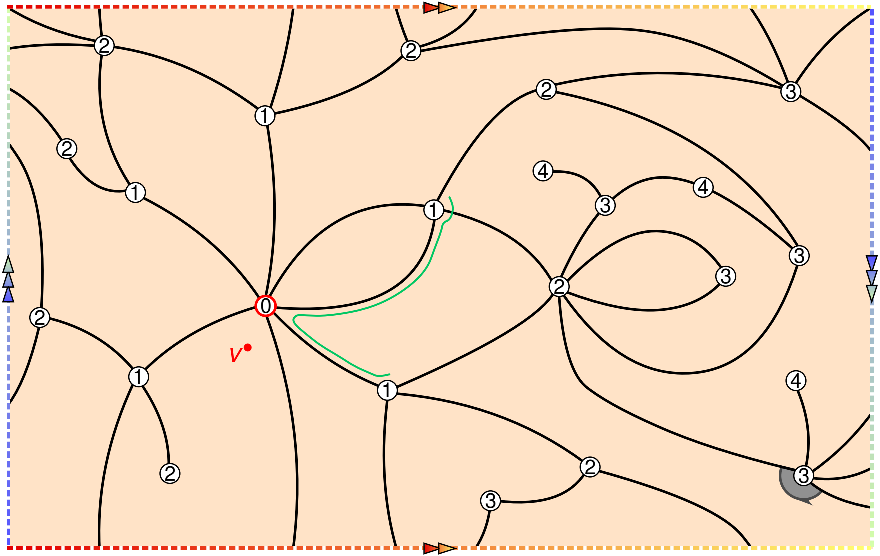
Level loops



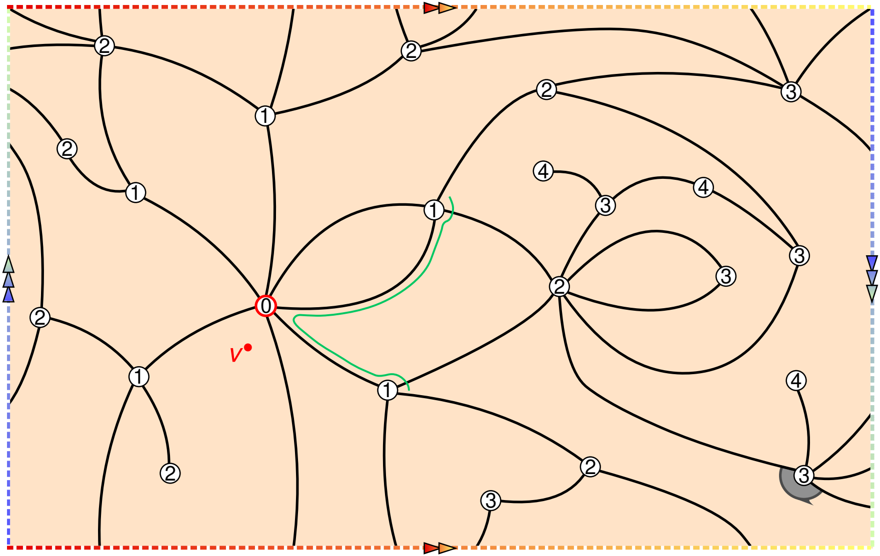
Level loops



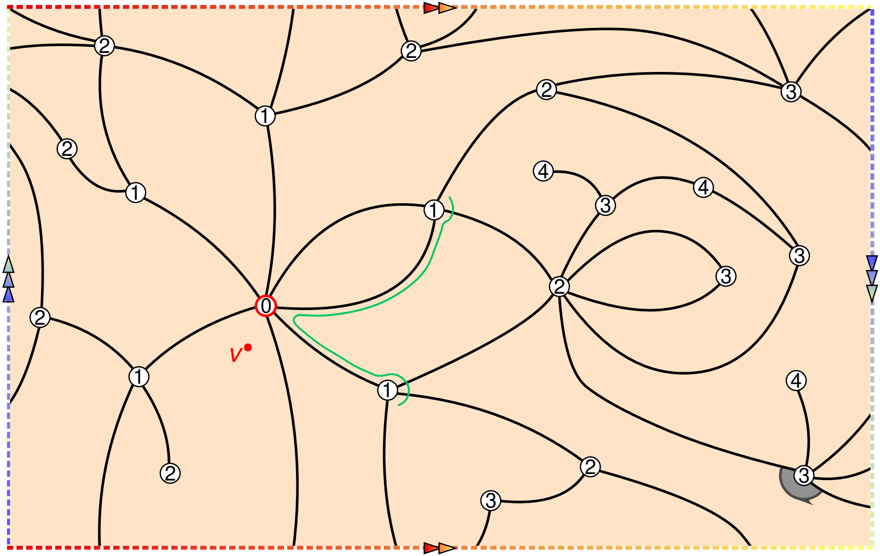
Level loops



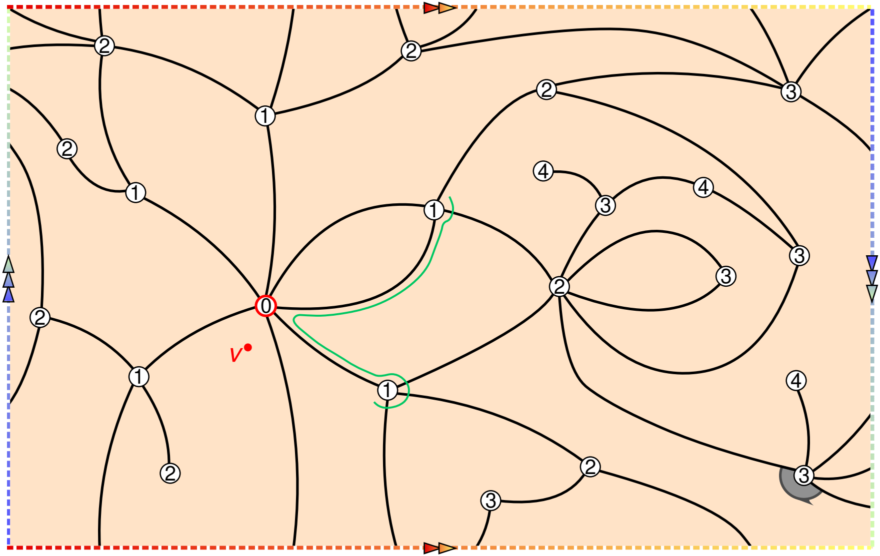
Level loops



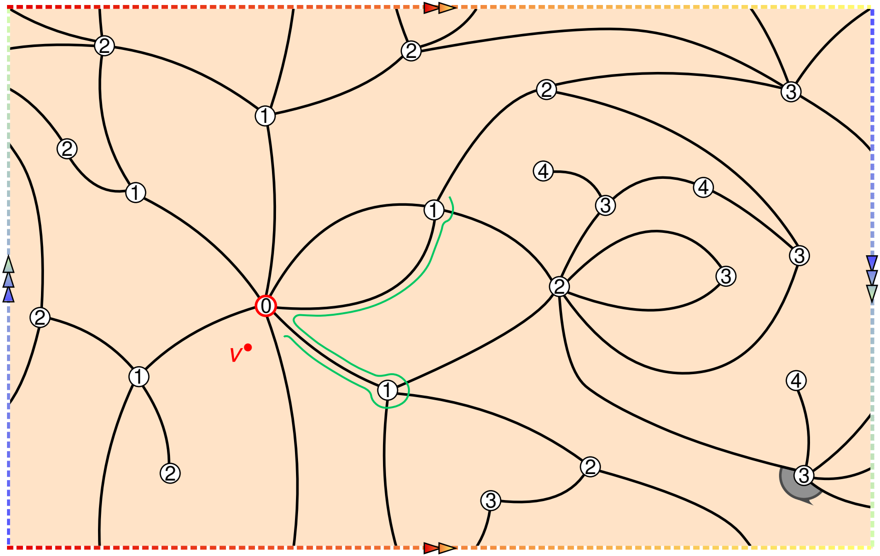
Level loops



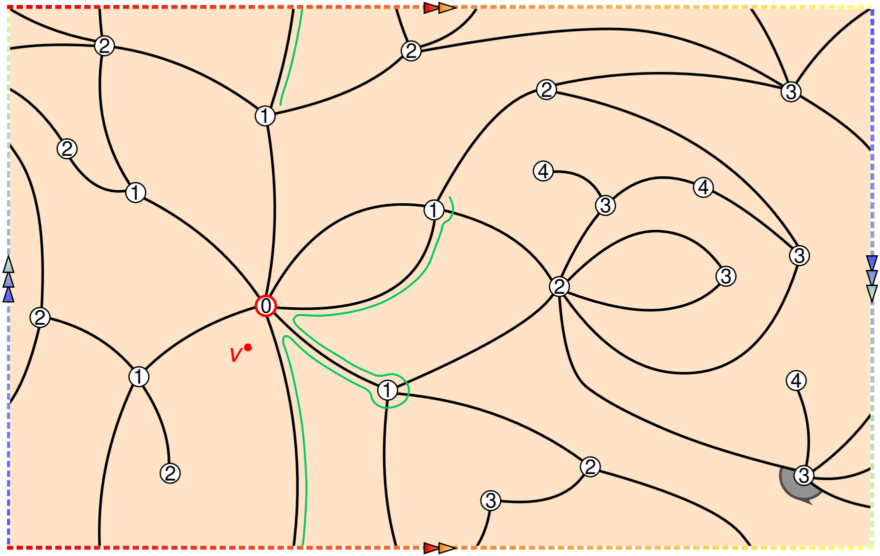
Level loops



Level loops



Level loops



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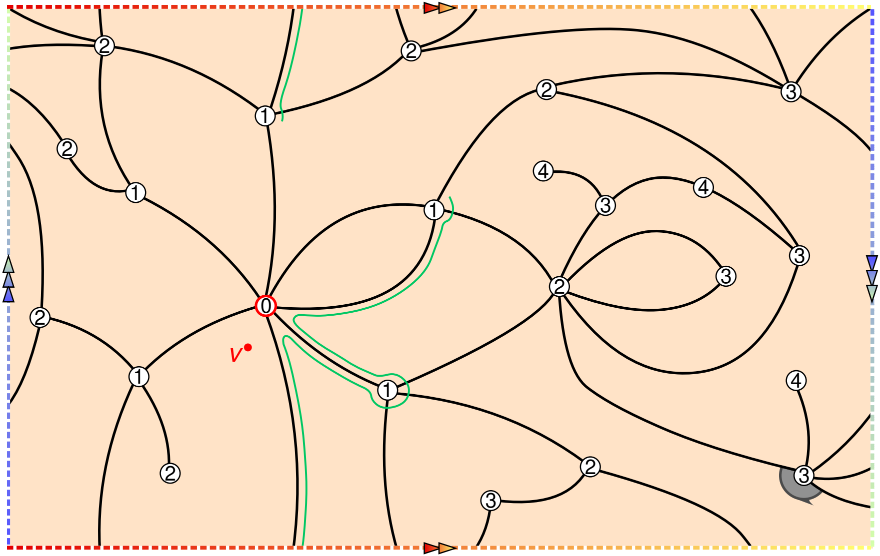

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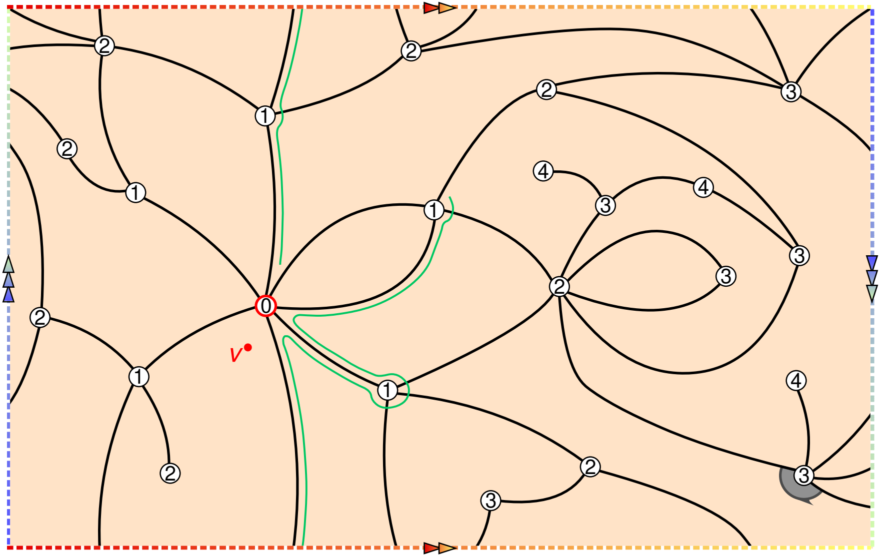

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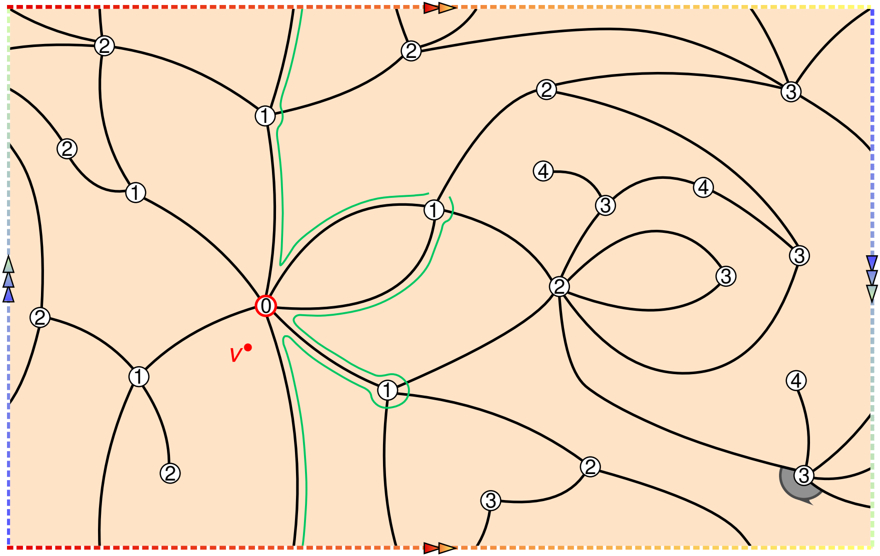
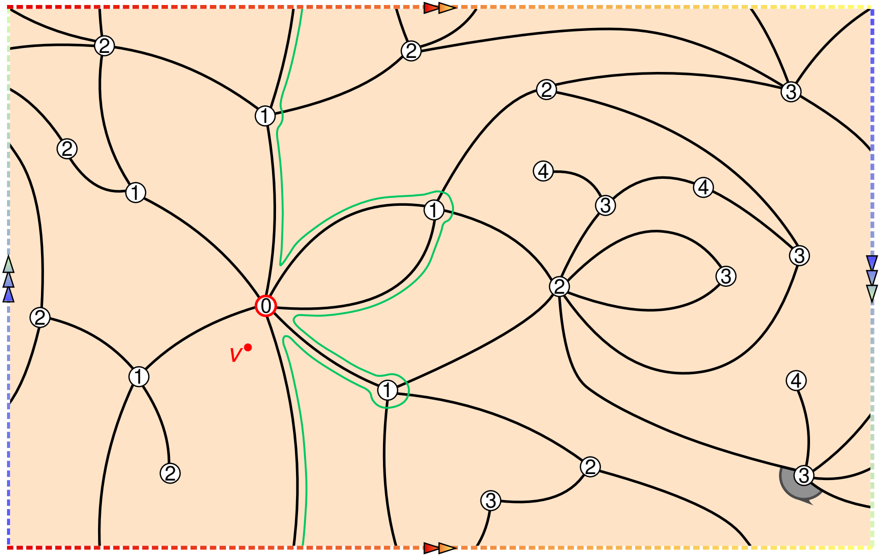


Level loops



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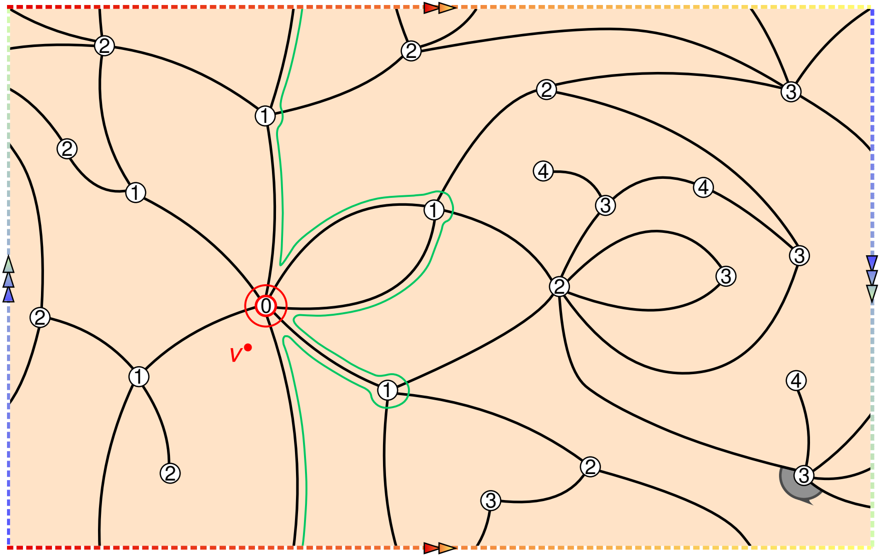

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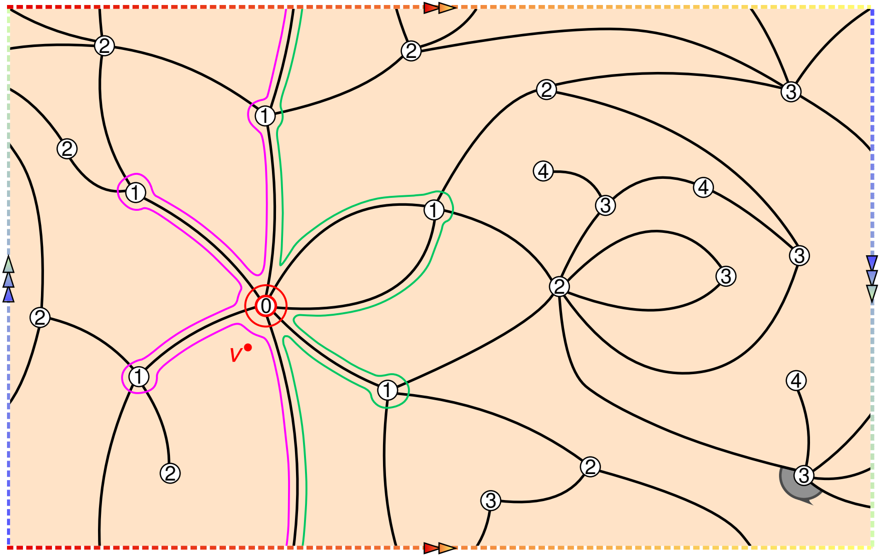
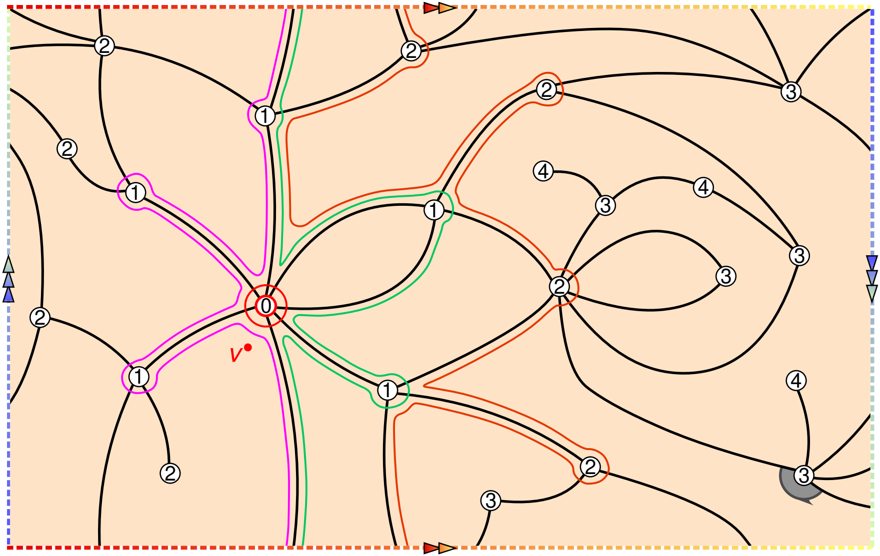
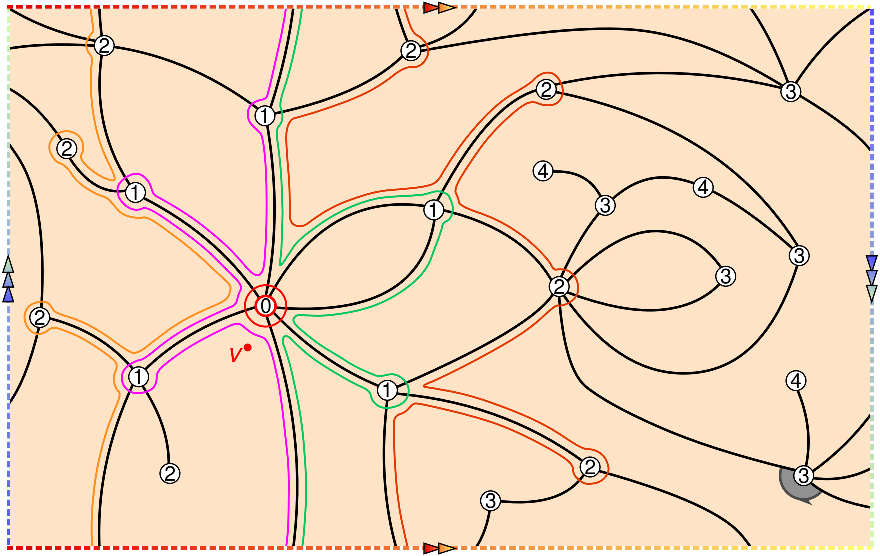


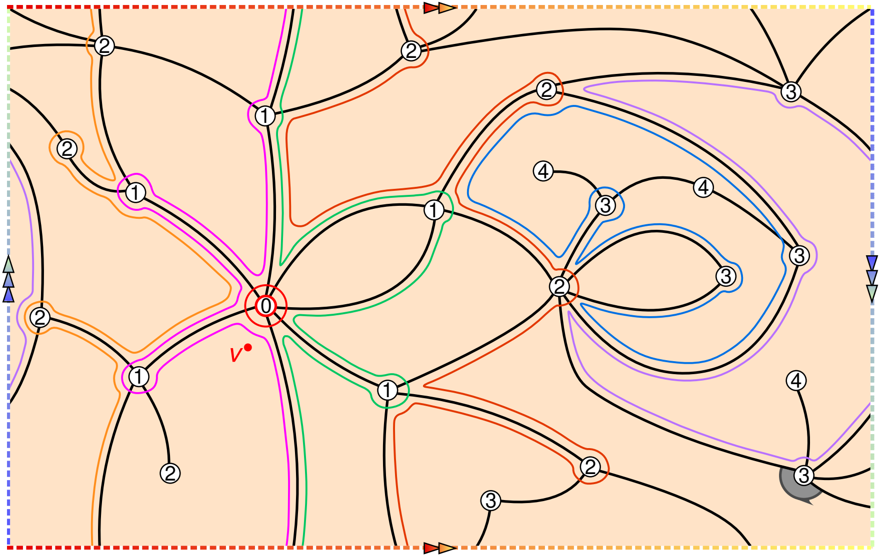
Level loops



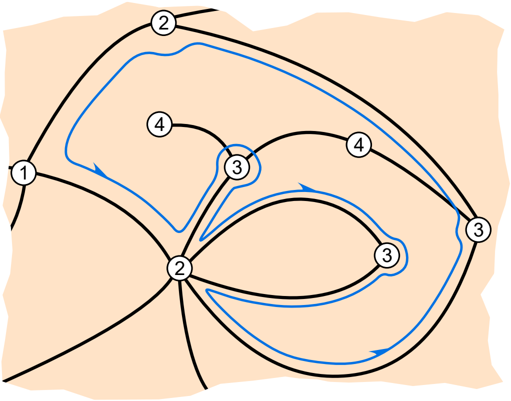
Level loops



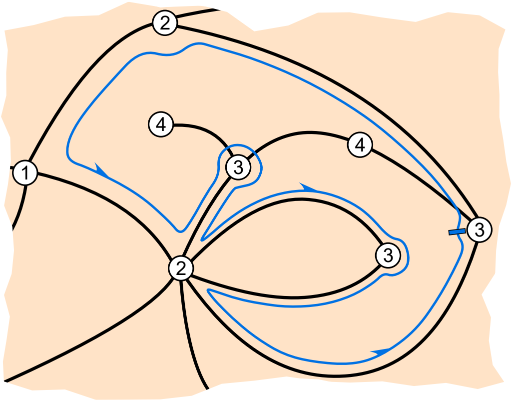
Level loops



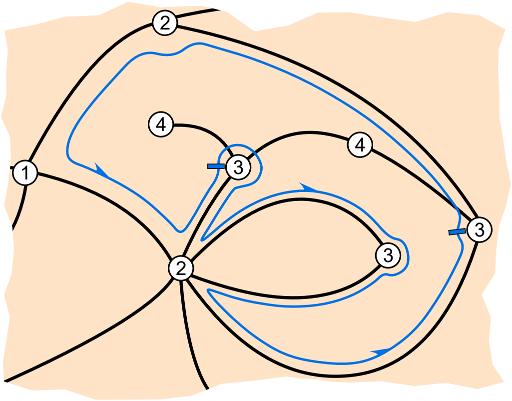
Definition of stops after orientation



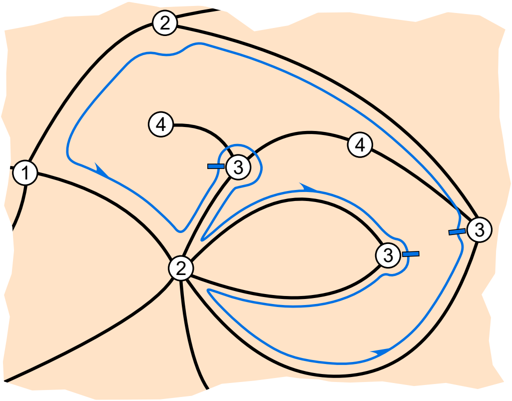
Definition of stops after orientation



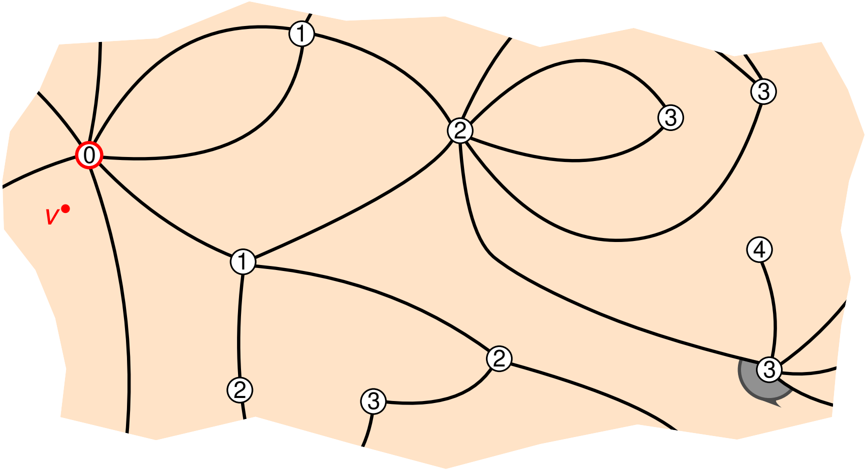
Definition of stops after orientation



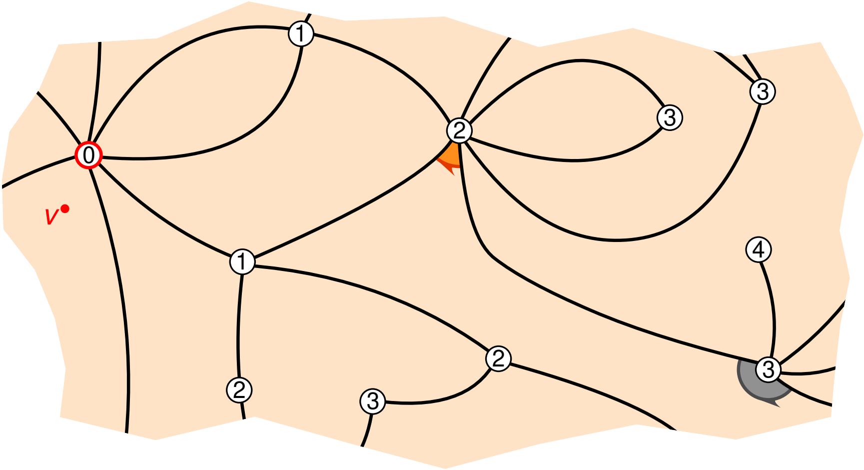
Definition of stops after orientation



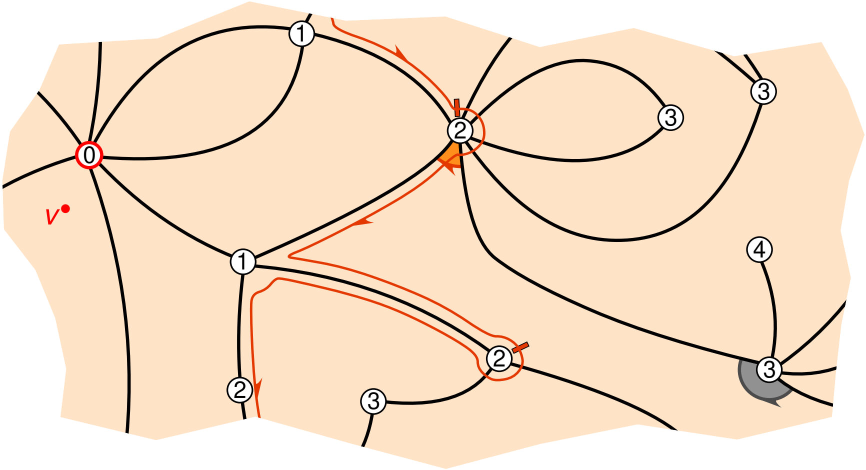
Orientation of the level loops: initialization



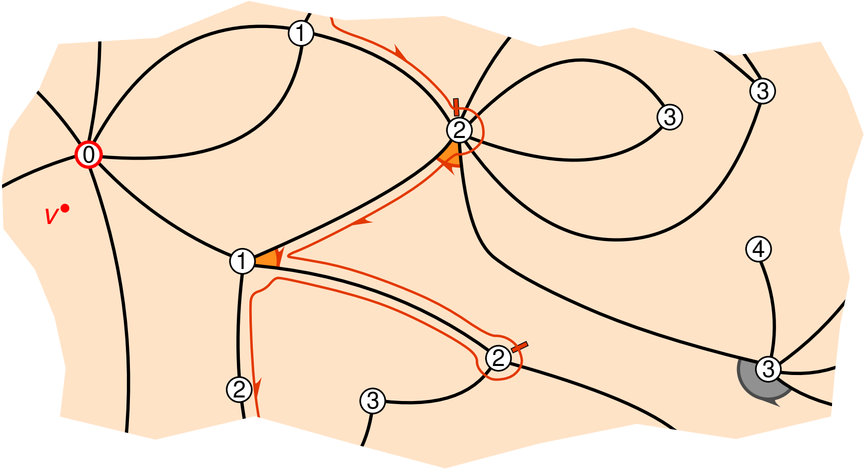
Orientation of the level loops: initialization



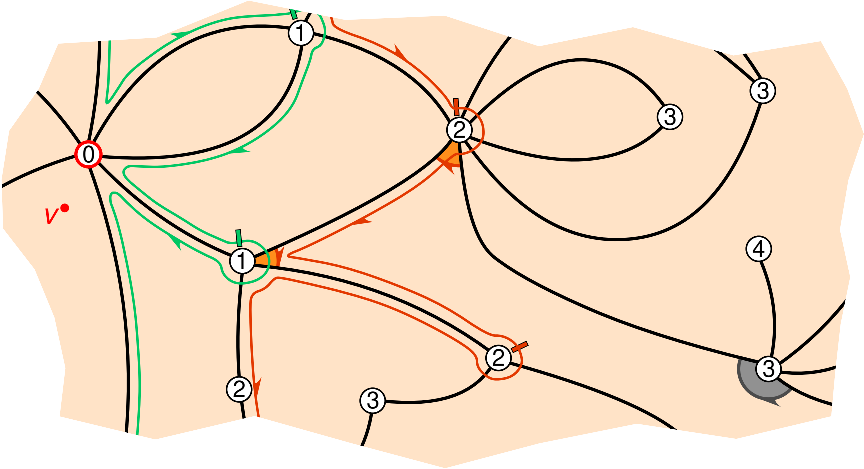
Orientation of the level loops: initialization



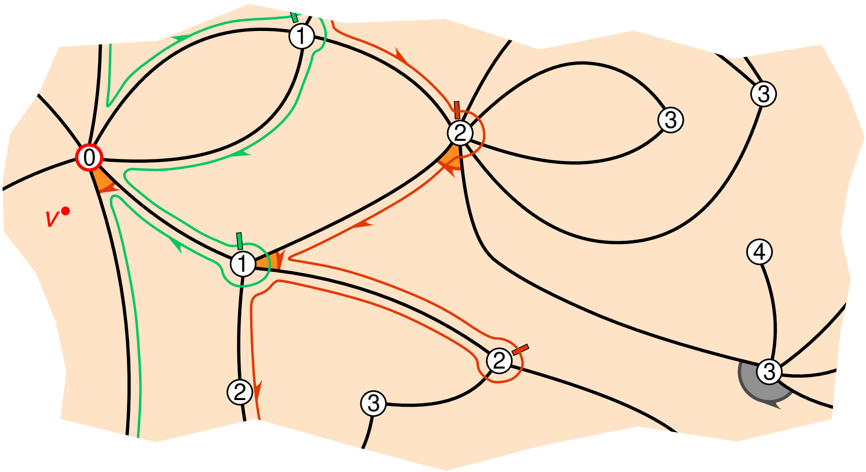
Orientation of the level loops: initialization



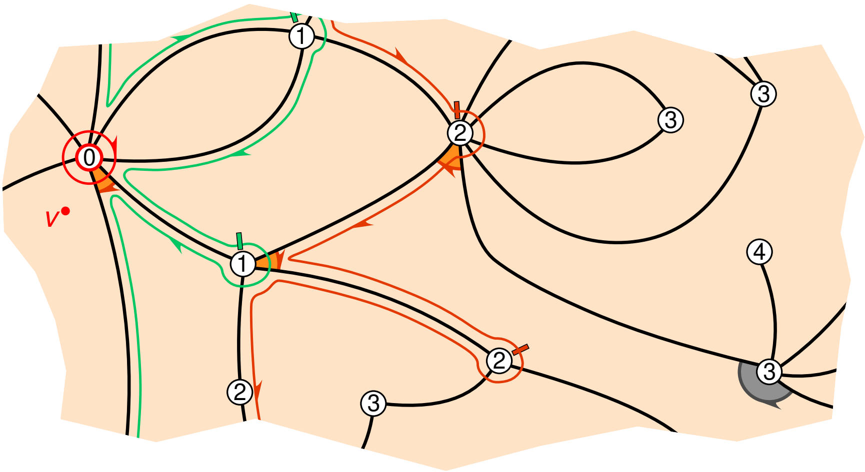
Orientation of the level loops: initialization



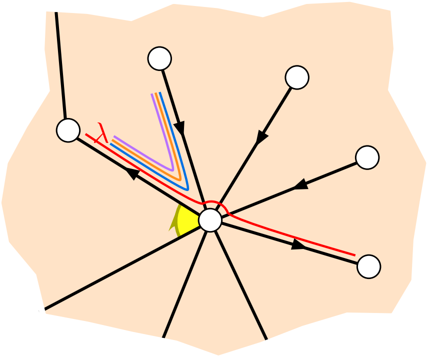
Orientation of the level loops: initialization



Orientation of the level loops: initialization

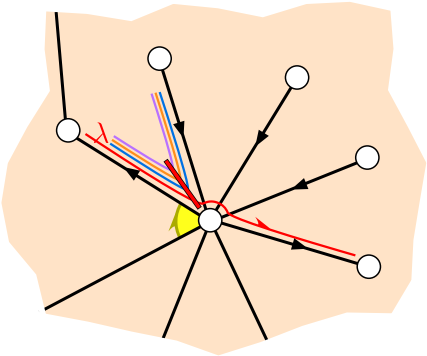


Orientation of the level loops: corner exploration



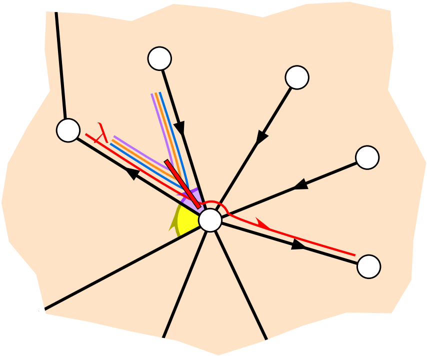
- *Turning around the initial vertex*

Orientation of the level loops: corner exploration



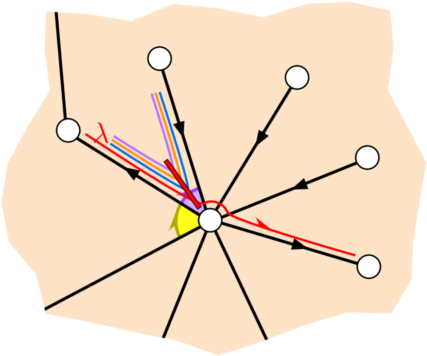
- *Turning around the initial vertex*
  - Orient λ if not oriented yet.

Orientation of the level loops: corner exploration



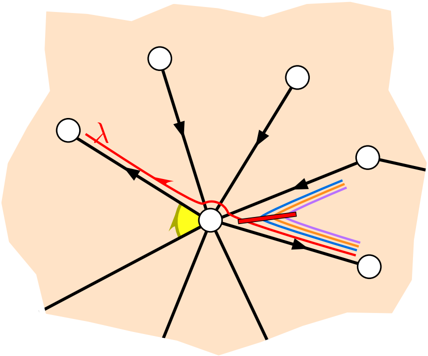
- *Turning around the initial vertex*
  - Orient λ if not oriented yet.
  - Turn to next stop.

Orientation of the level loops: corner exploration



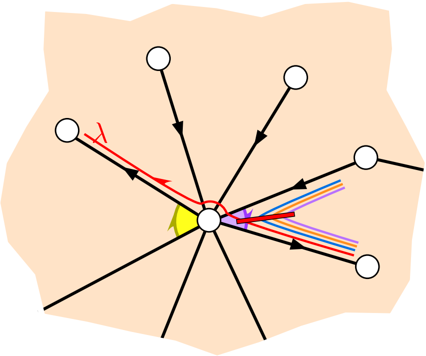
- *Turning around the initial vertex*
  - Orient λ if not oriented yet.
  - Turn to next stop.
  - Orient all the nonoriented loops.

Orientation of the level loops: corner exploration



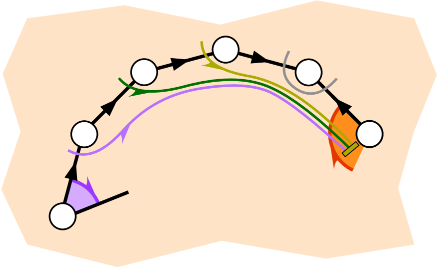
- *Turning around the initial vertex*
  - Orient λ if not oriented yet.
  - Turn to next stop.
  - Orient all the nonoriented loops.

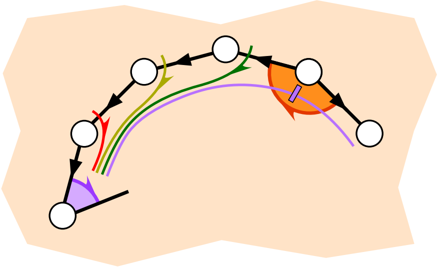
Orientation of the level loops: corner exploration



- *Turning around the initial vertex*
  - Orient λ if not oriented yet.
  - Turn to next stop.
  - Orient all the nonoriented loops.

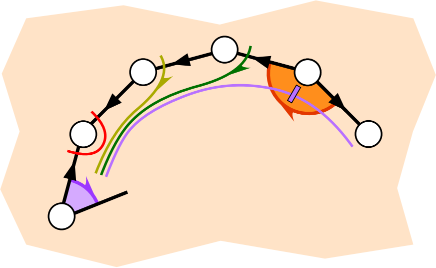
Orientation of the level loops: corner exploration

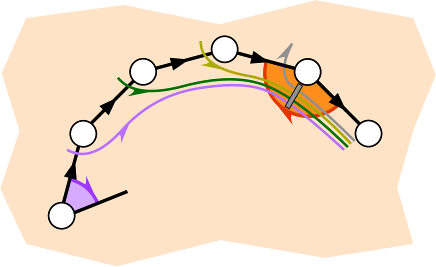




- *Turning around the initial vertex*
  - Orient λ if not oriented yet.
  - Turn to next stop.
  - Orient all the nonoriented loops.
- *Moving around the face*
  - until there is a stop
  - or orienting the loop creates a stop.

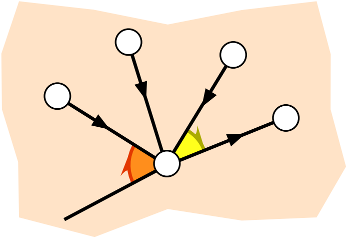
Orientation of the level loops: corner exploration





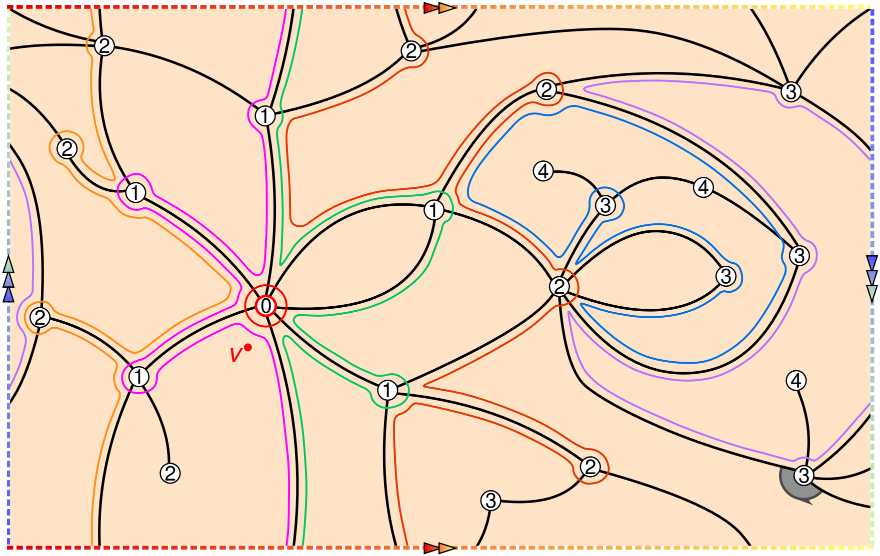
- *Turning around the initial vertex*
  - Orient λ if not oriented yet.
  - Turn to next stop.
  - Orient all the nonoriented loops.
- *Moving around the face*
  - until there is a stop
  - or orienting the loop creates a stop.

Orientation of the level loops: corner exploration

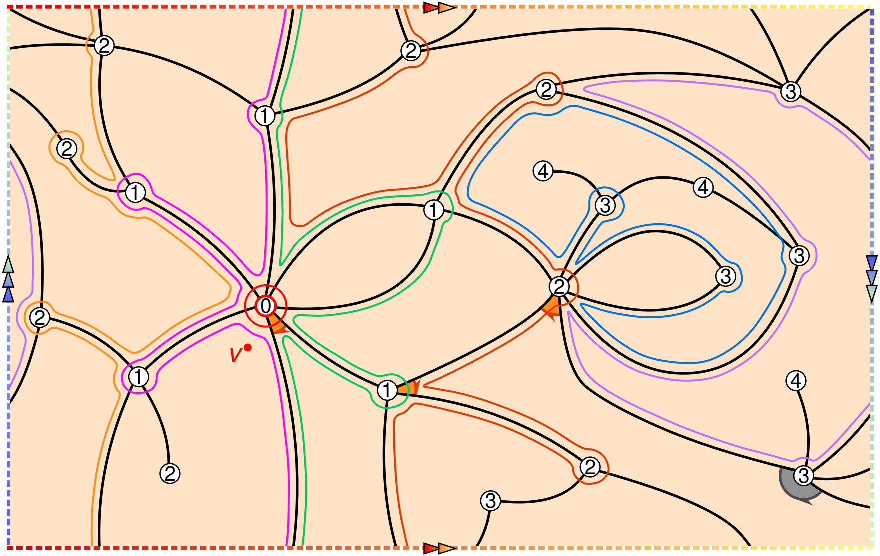


- *Turning around the initial vertex*
  - Orient λ if not oriented yet.
  - Turn to next stop.
  - Orient all the nonoriented loops.
- *Moving around the face*
  - until there is a stop
  - or orienting the loop creates a stop.
- *Turning around the final vertex*
  - Turn to next outgoing edge.

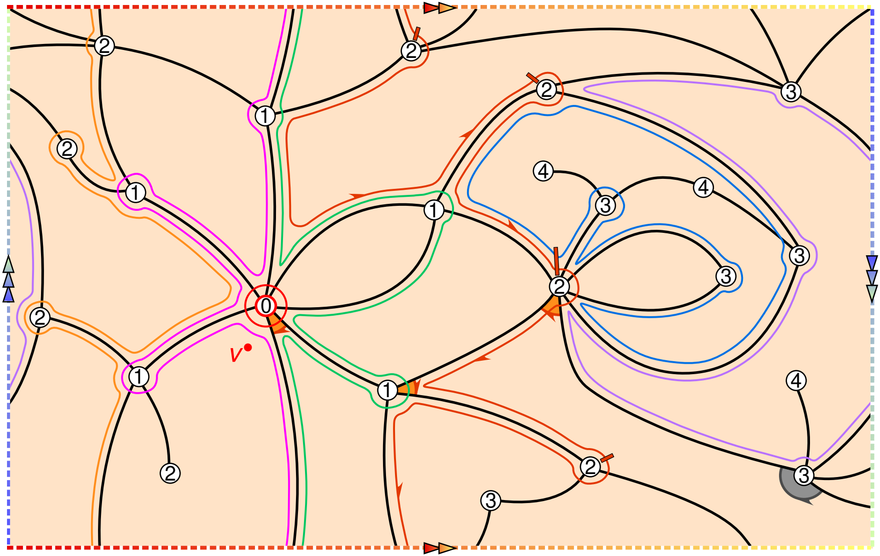
From pointed bipartite maps to unicellular mobiles



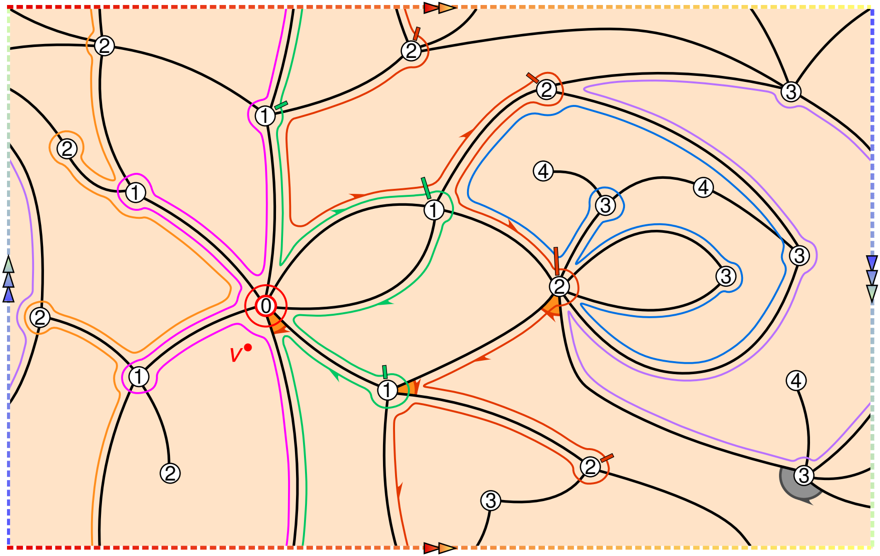
From pointed bipartite maps to unicellular mobiles



From pointed bipartite maps to unicellular mobiles



From pointed bipartite maps to unicellular mobiles



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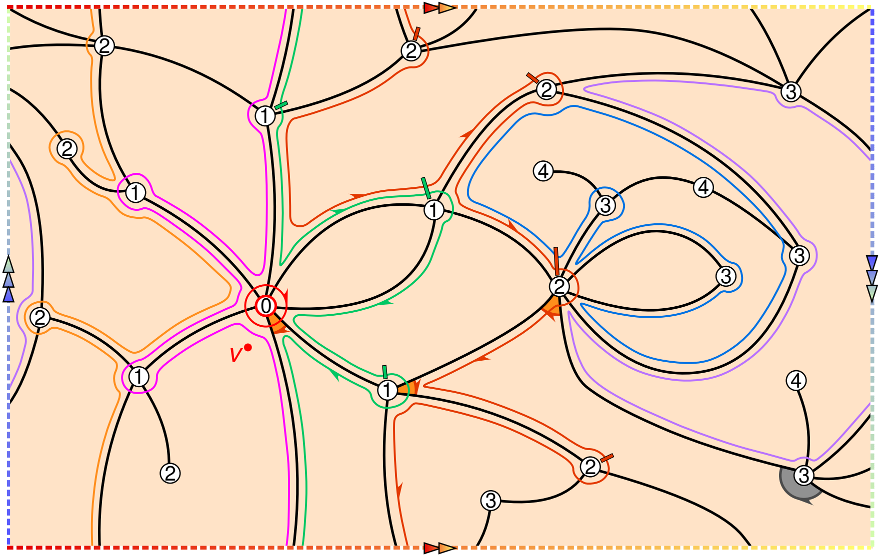

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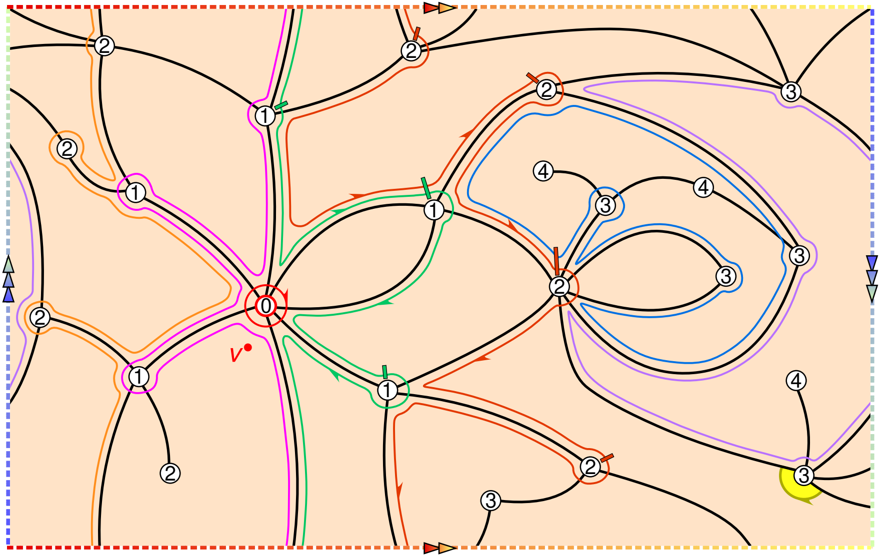

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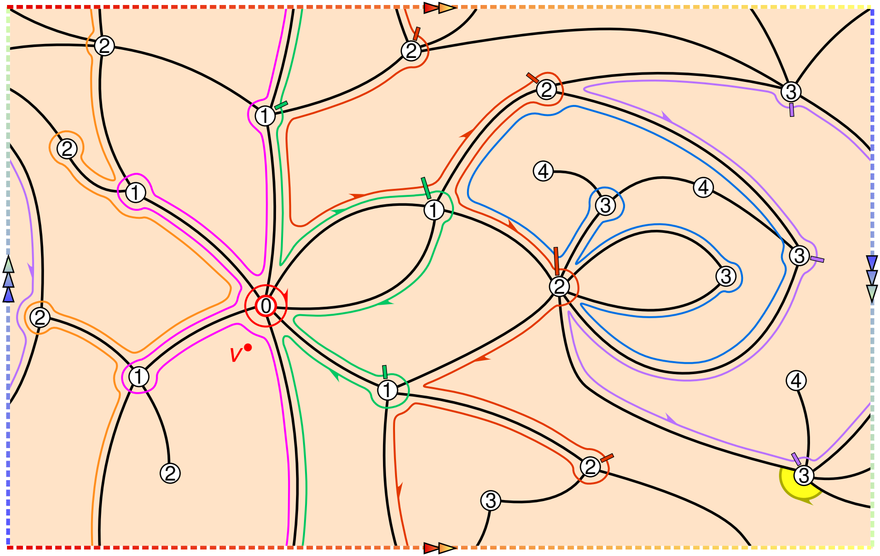

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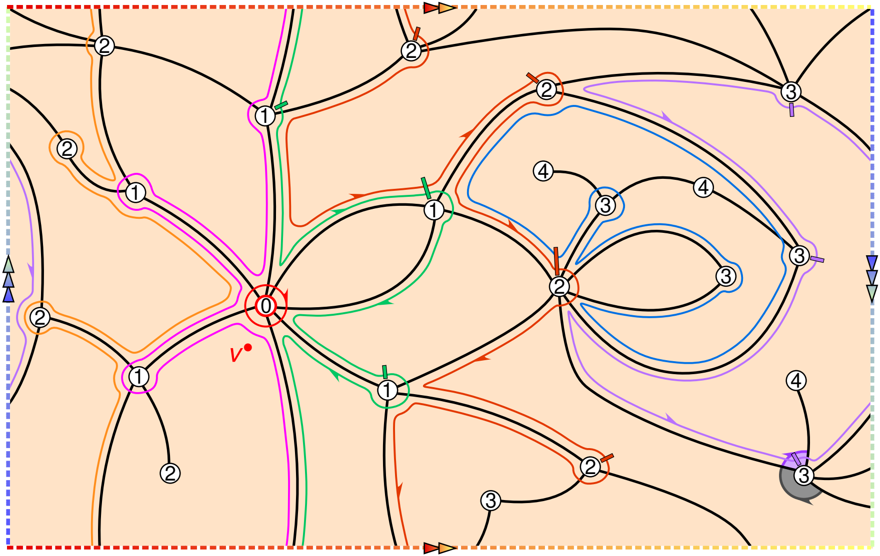

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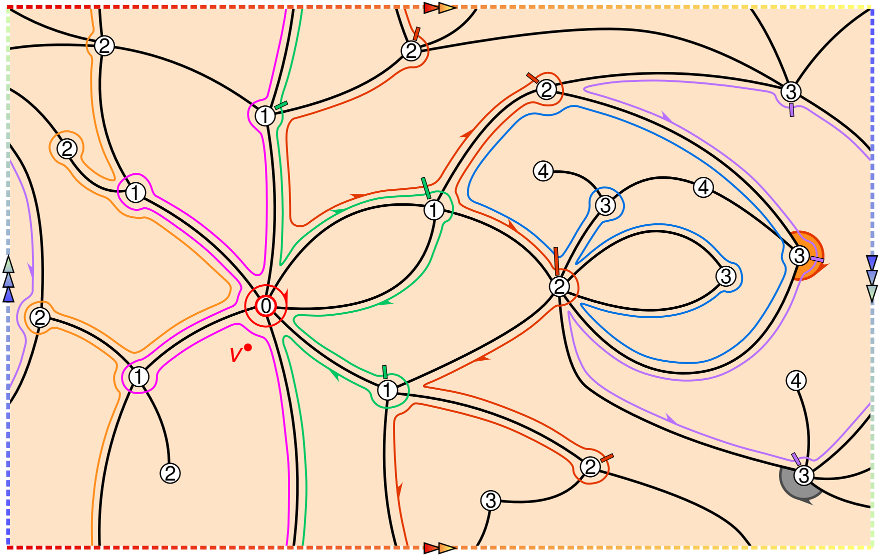

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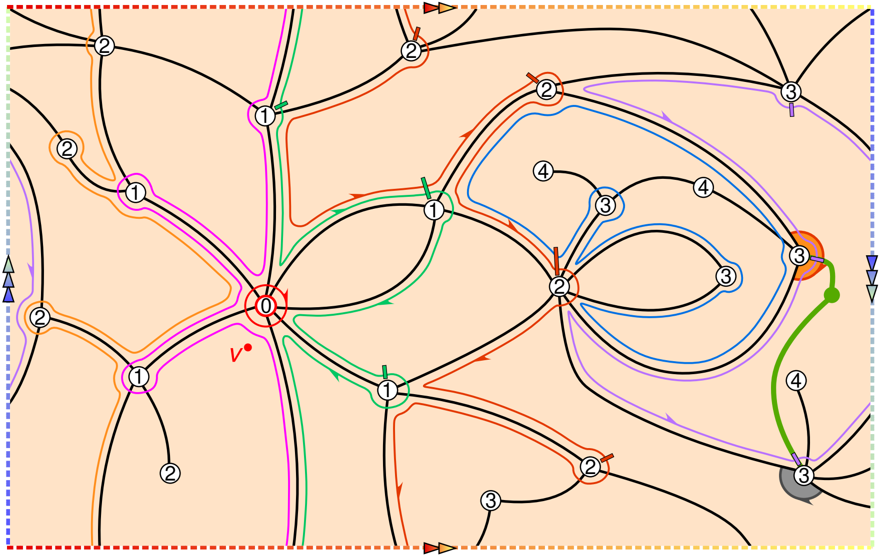

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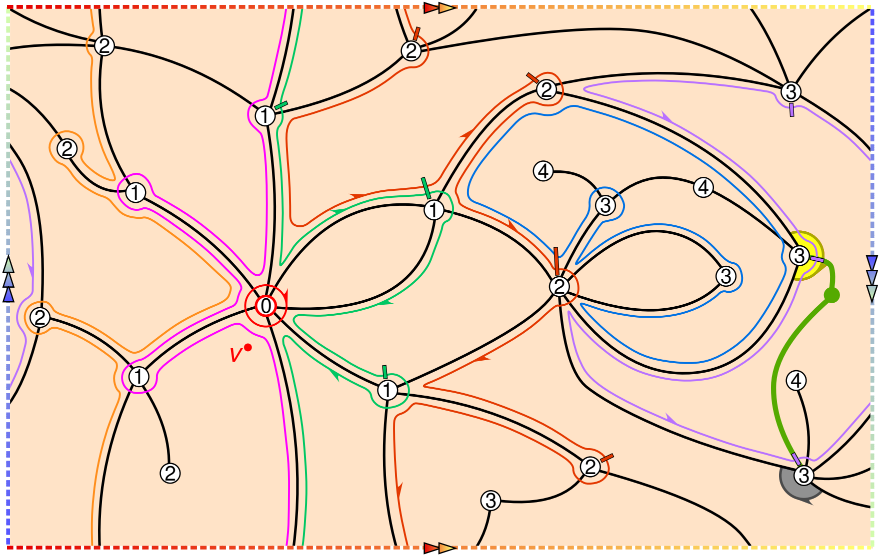

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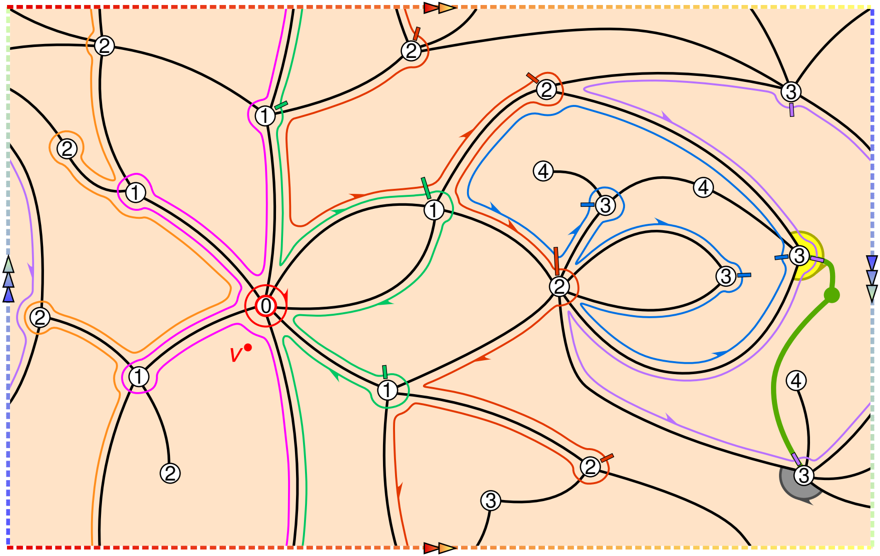

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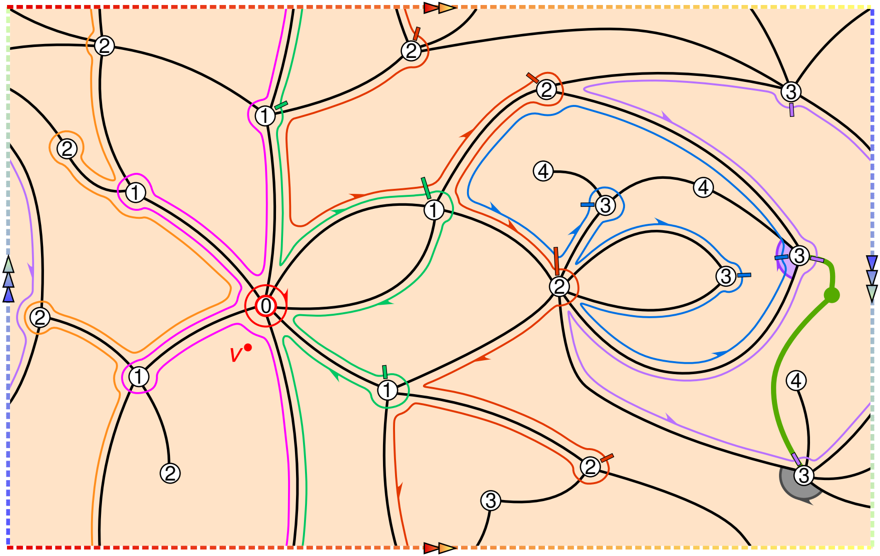

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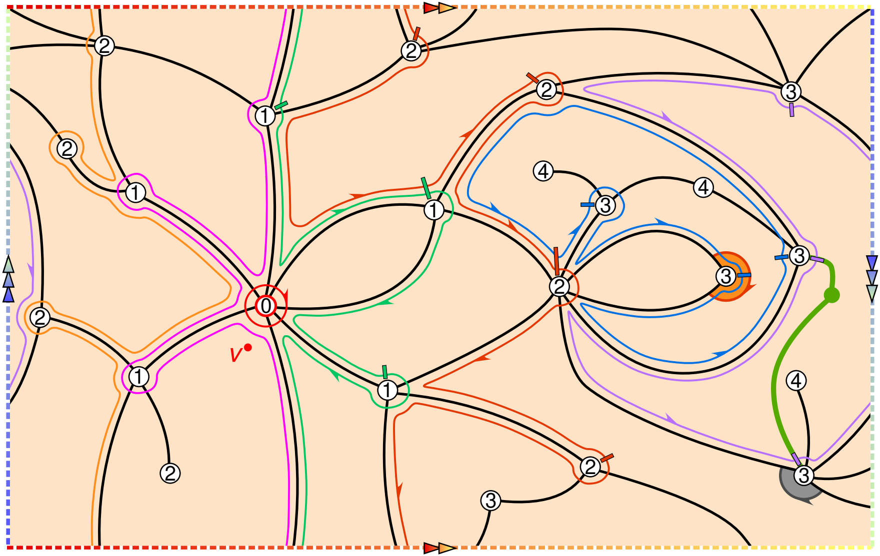

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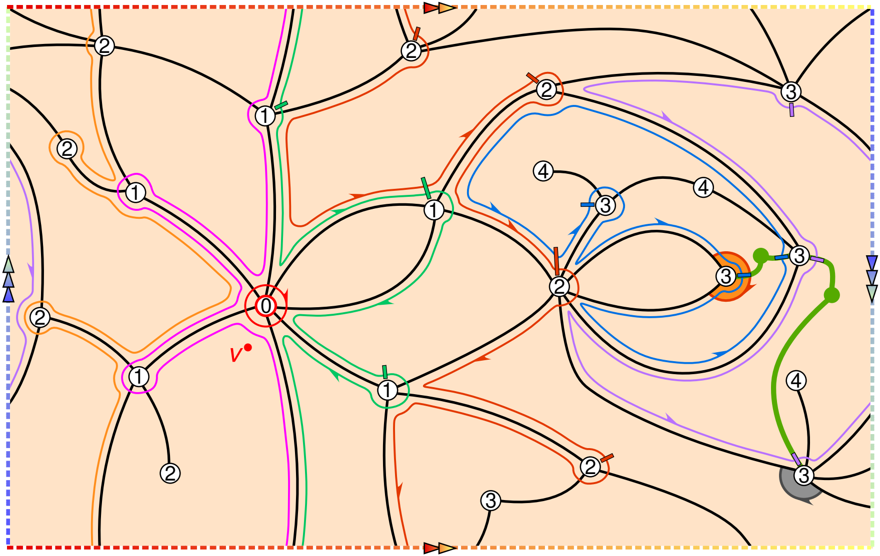

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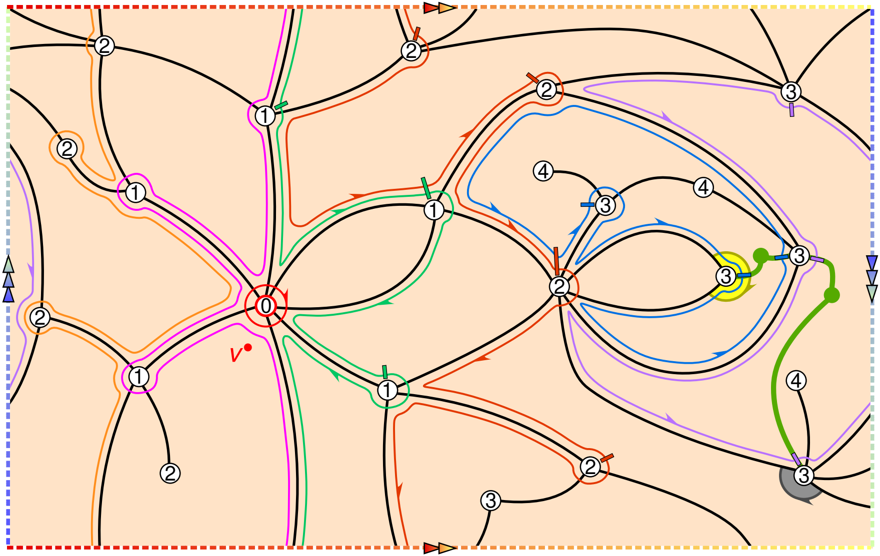

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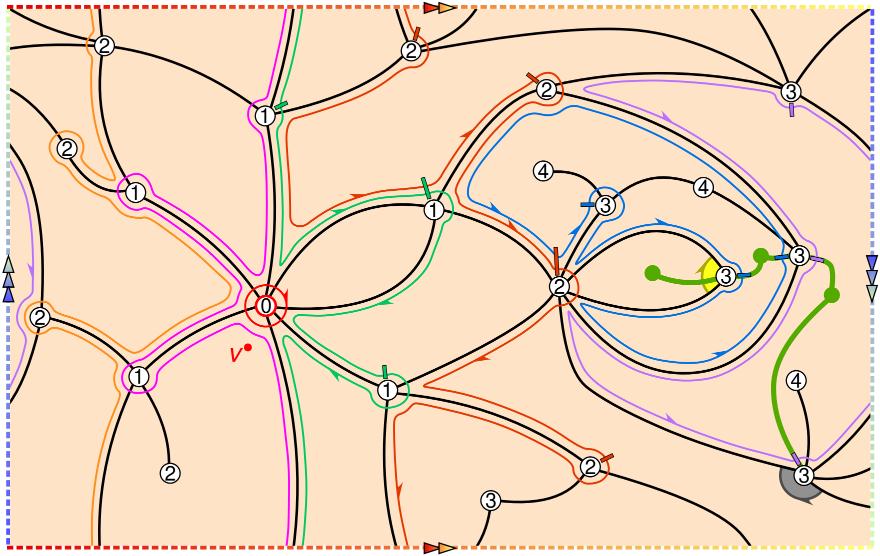

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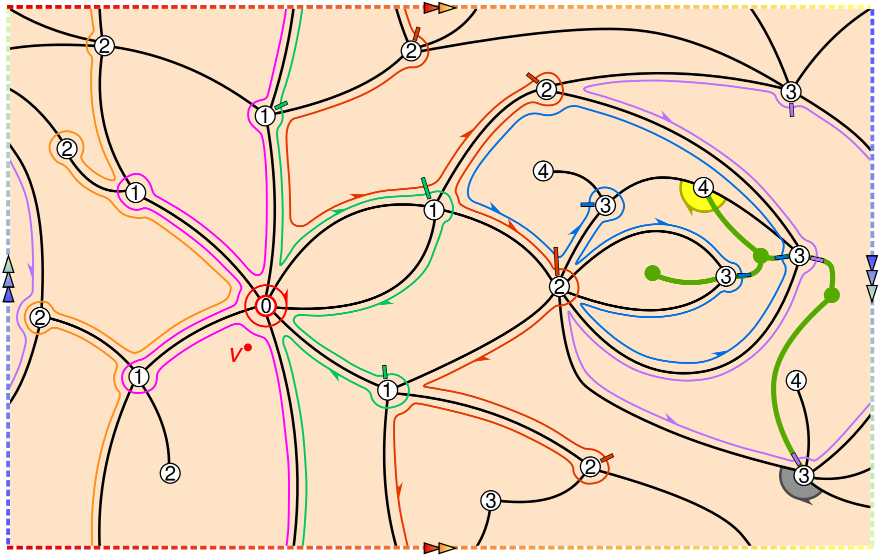

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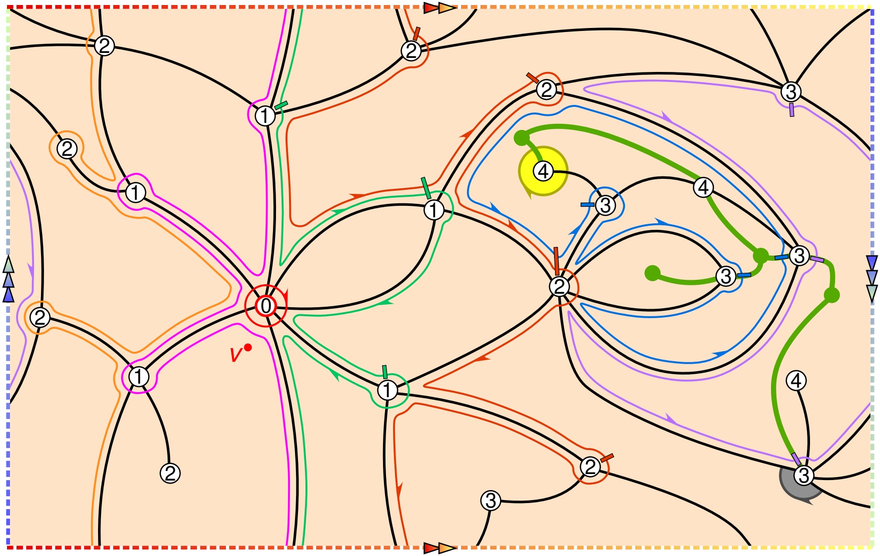

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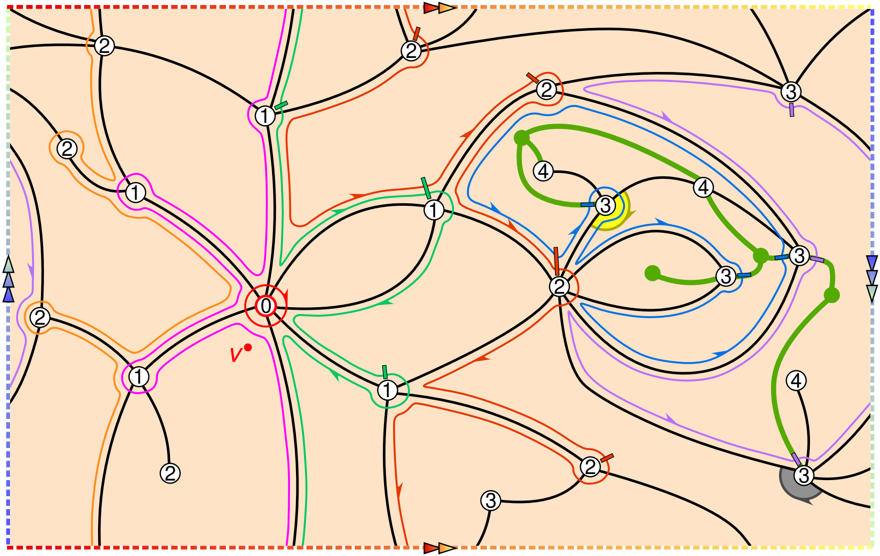

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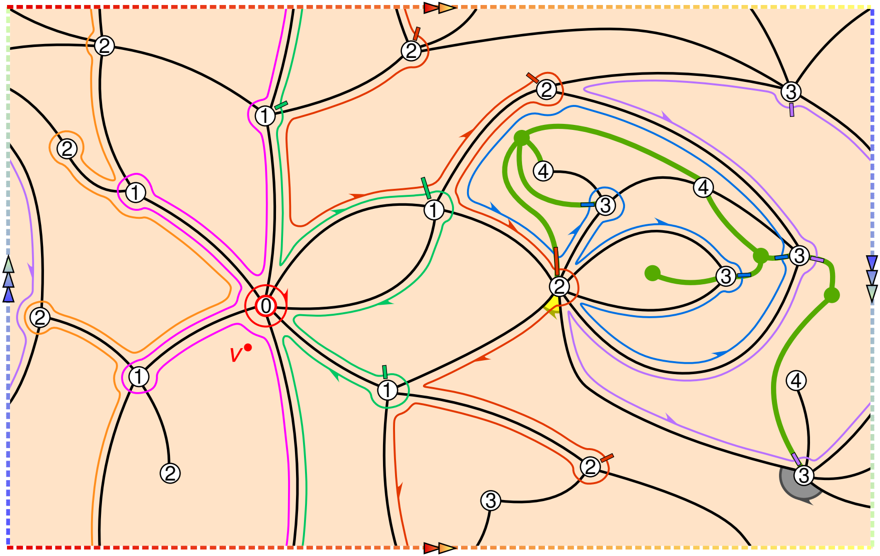

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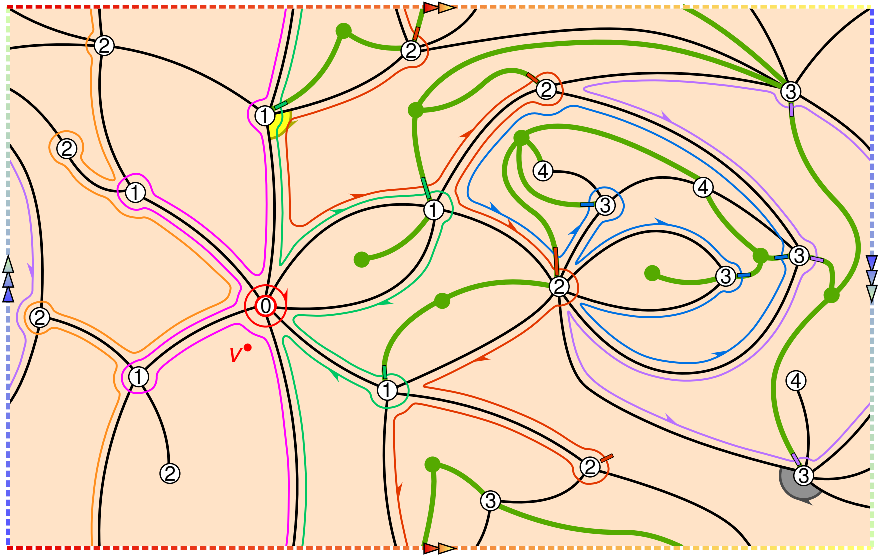

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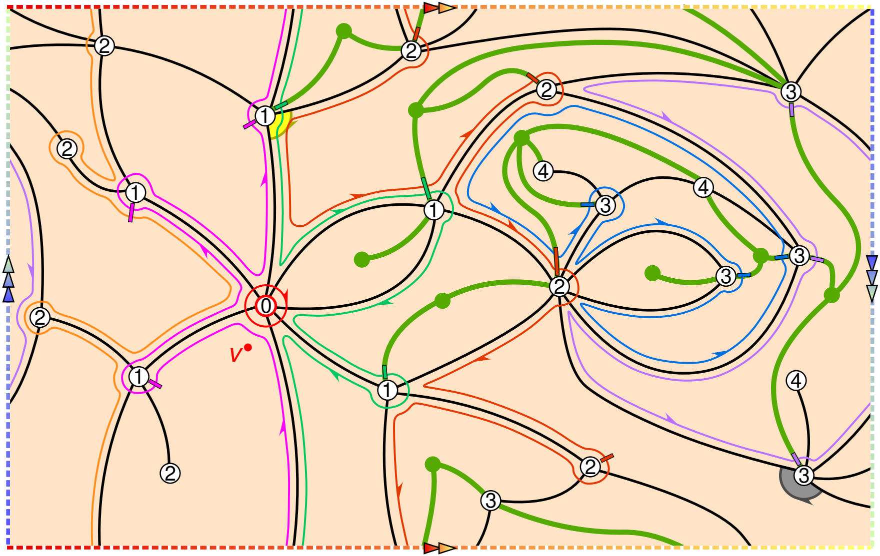

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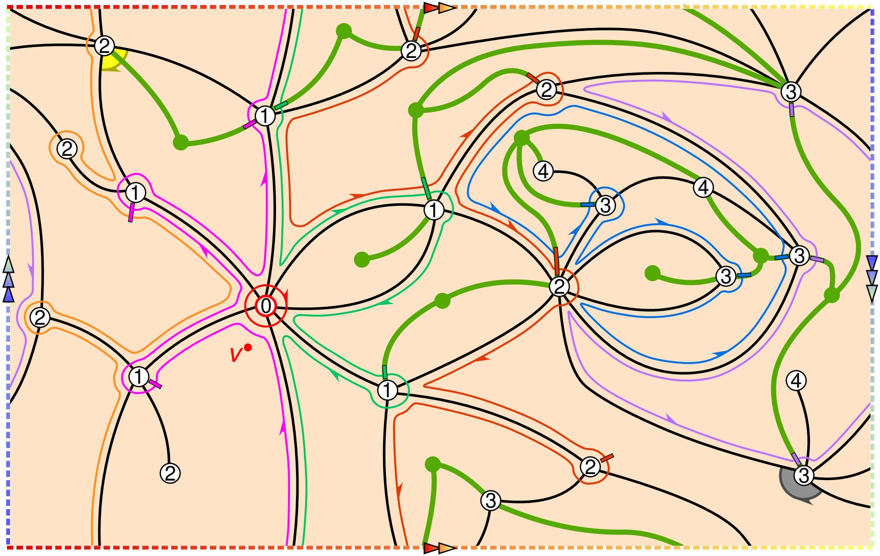

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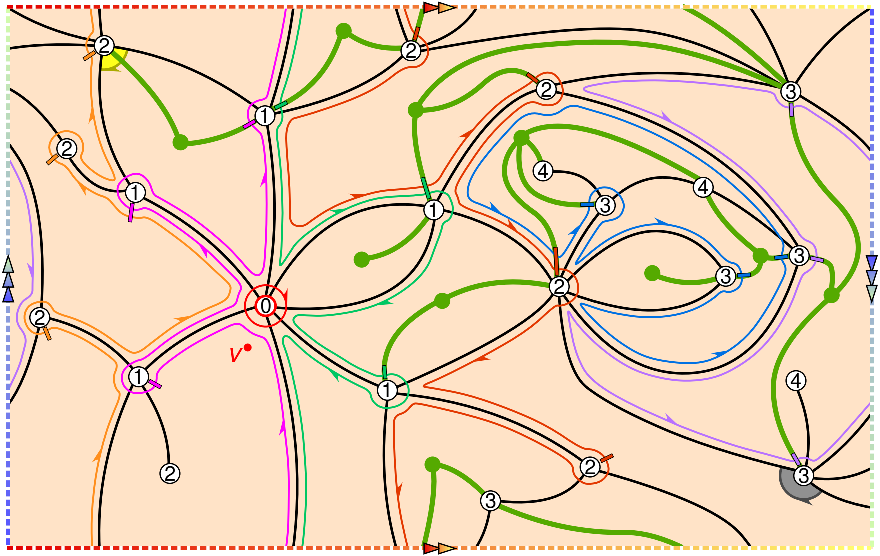

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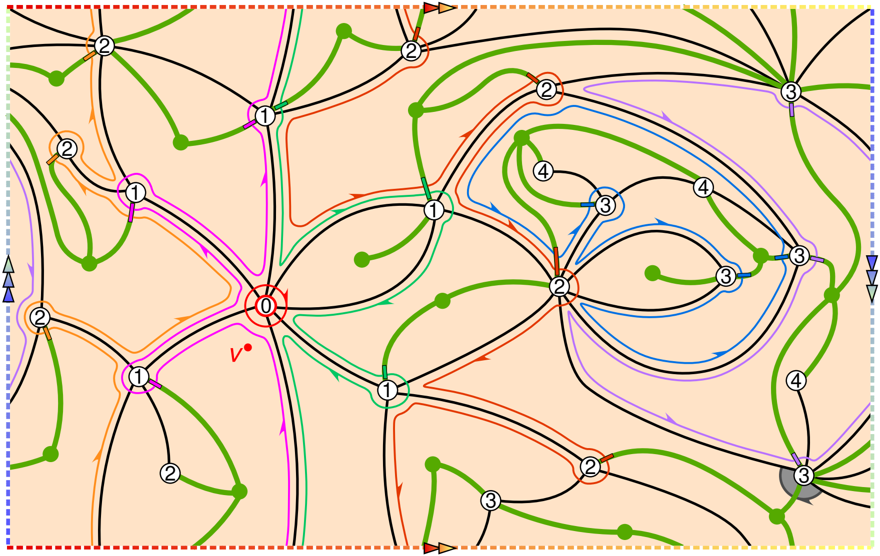

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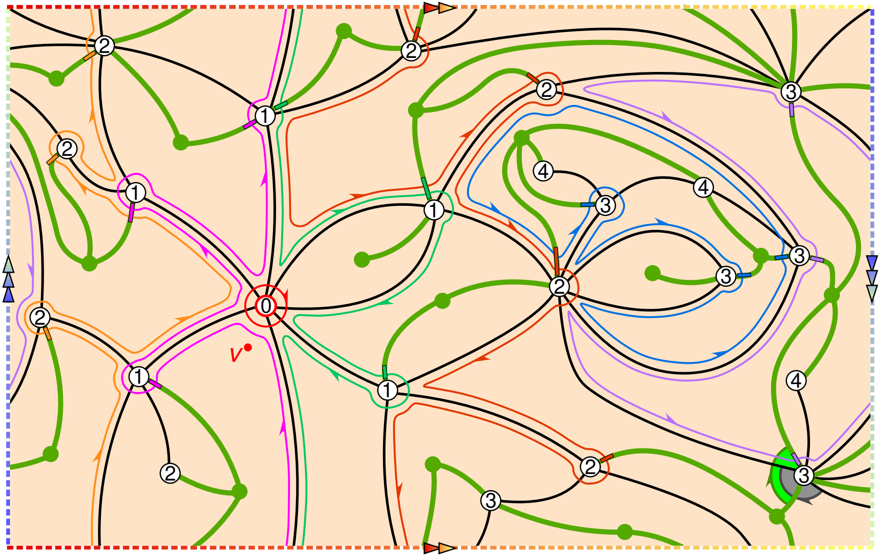

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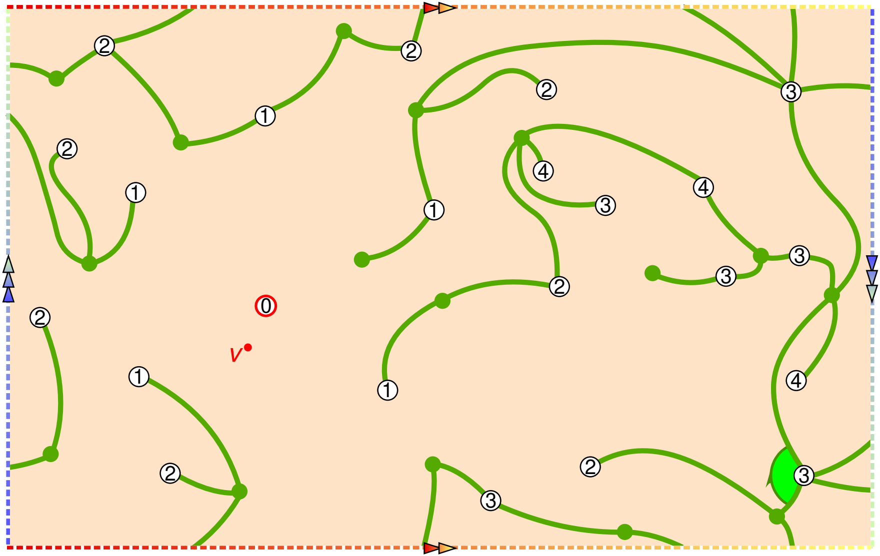


Corresponding quantities

Proposition

Let $(\mathfrak{m}, v^\bullet)$ be a pointed bipartite map and $(\mathfrak{u}, \mathfrak{l})$ the corresponding well-labeled unicellular mobile. Then

- $V(\mathfrak{m}) = V_\circ(\mathfrak{u}) \sqcup \{v^\bullet\}$ *and, for* $v \in V_\circ(\mathfrak{u})$, $\mathfrak{l}(v) = d_\mathfrak{m}(v, v^\bullet)$ *;*
- *the faces of* $\mathfrak{m}$ *correspond to* $V_\bullet(\mathfrak{u})$ *: moreover, the degree of a face of* $\mathfrak{m}$ *is twice the degree of the corresponding vertex in* $V_\bullet(\mathfrak{u})$*;*
- *the maps* $\mathfrak{m}$ *and* $\mathfrak{u}$ *have the same number of edges;*
- *among the edges of* $\mathfrak{m}$ *in a white corner of* $\mathfrak{u}$*, exactly one is directed toward* $v^\bullet$*.*

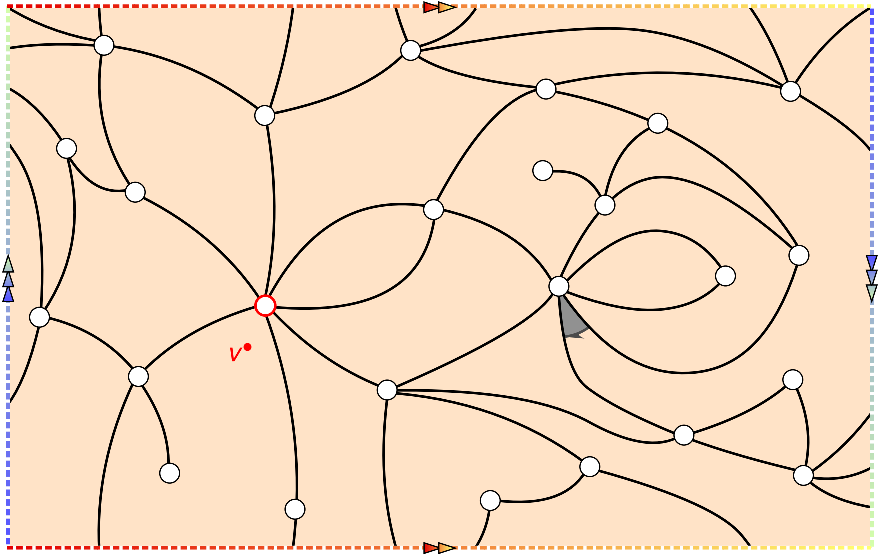
Degree precriptions

Specializations

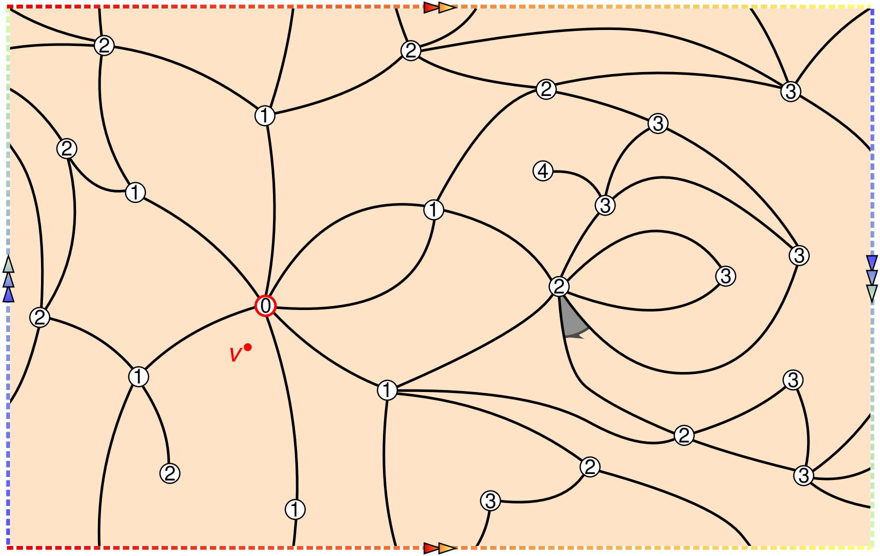
The previous construction specializes into a bijection between

- bipartite maps with n faces marked $1, 2, \ldots, n$ and such that, for $1 \leq i \leq n$, the face marked i has degree $2\alpha_i$;
- pairs of a sign $+$ or $-$ and a well-labeled unicellular mobile with n green vertices marked $1, 2, \ldots, n$ and such that, for $1 \leq i \leq n$, the green vertex marked i has degree α_i.

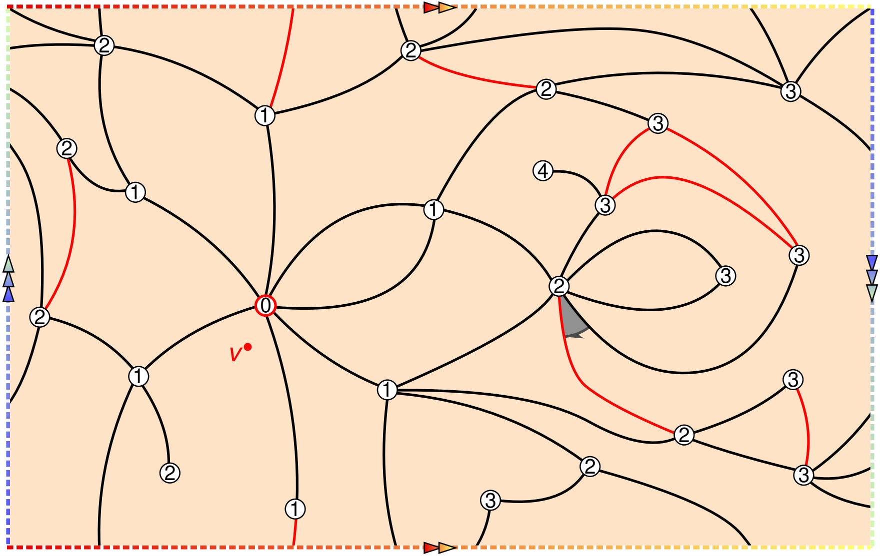
The construction for nonbipartite maps



The construction for nonbipartite maps



The construction for nonbipartite maps



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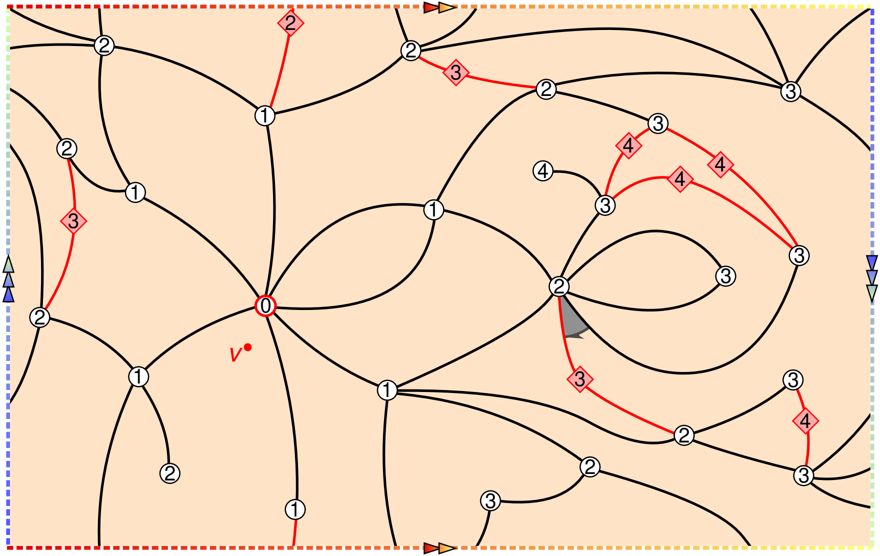

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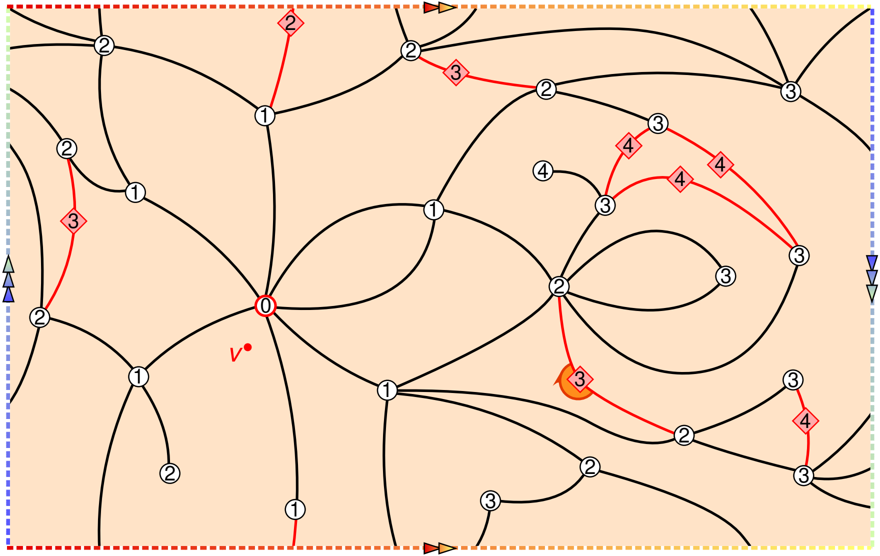

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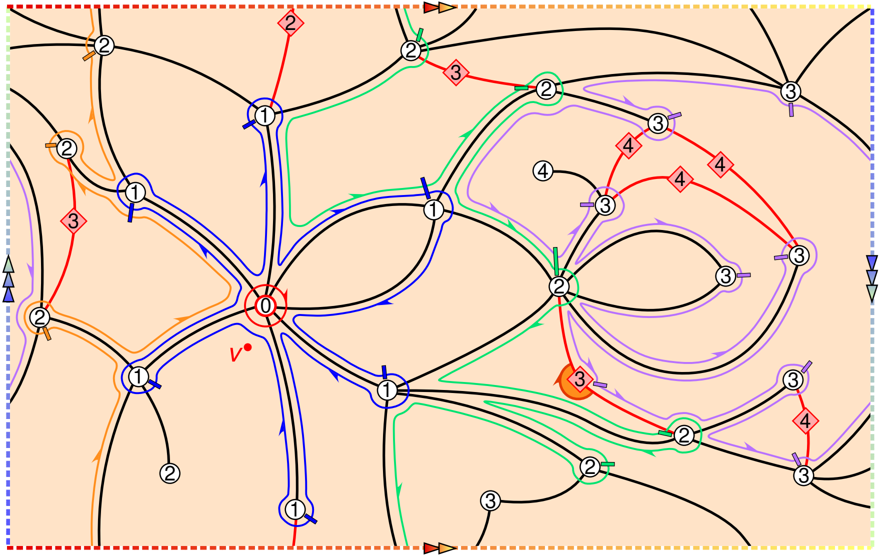

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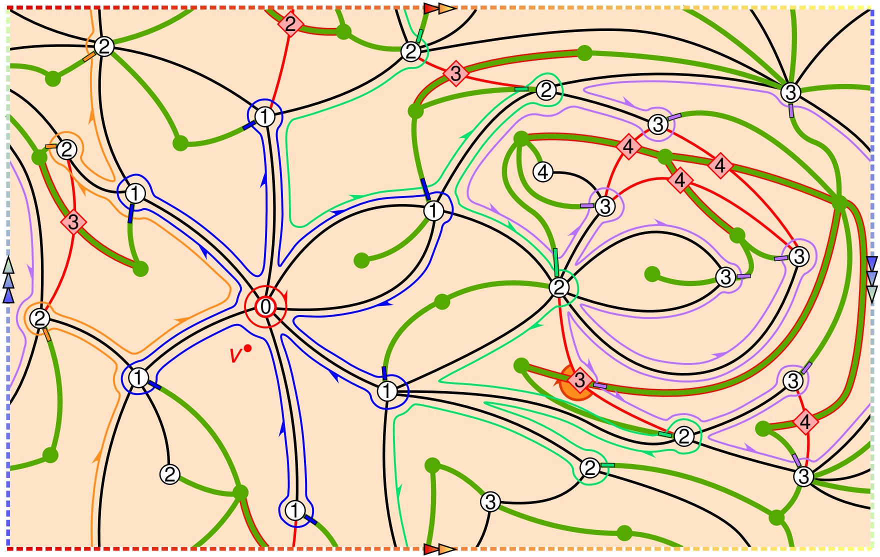

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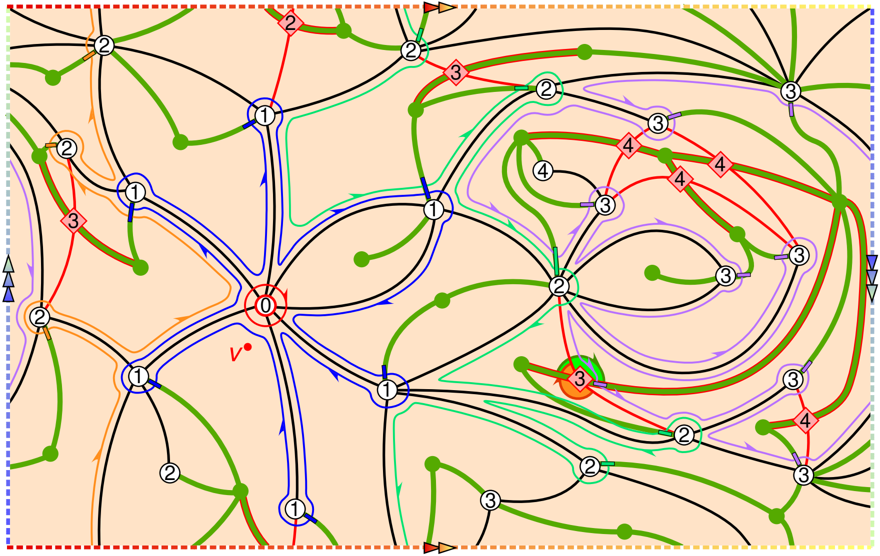

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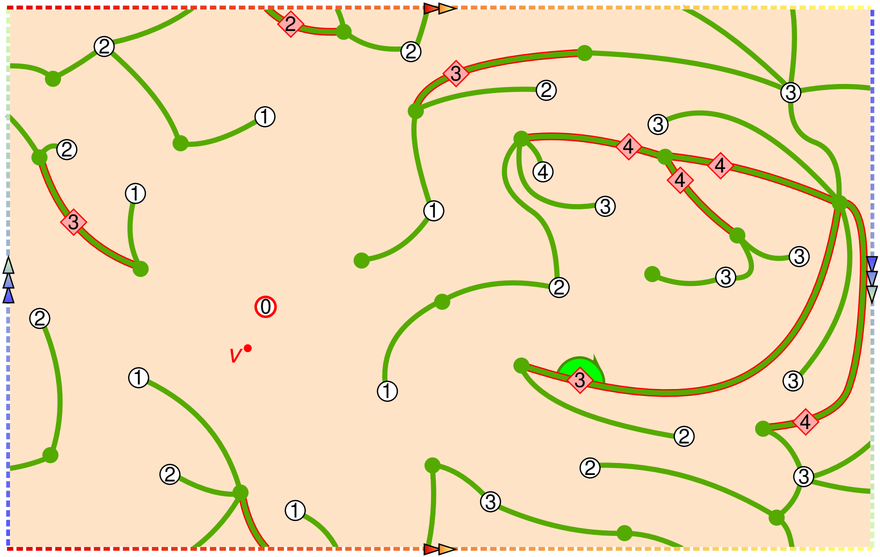


Well-labeled unicellular maps

Proposition

A labeled unicellular mobile whose green vertices all have degree 2 is well labeled iff the labels of neighboring vertices differ by at most 1.

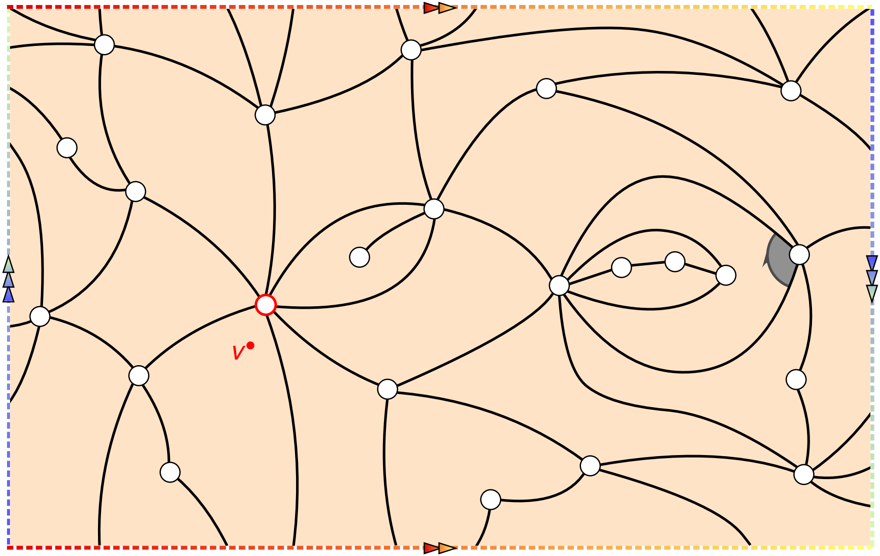
Well-labeled unicellular maps

Proposition

A labeled unicellular mobile whose green vertices all have degree 2 *is well labeled iff the labels of neighboring vertices differ by at most* 1*.*

In fact, no matter how we orient the corners in the definition of successors, a labeled unicellular mobile whose green vertices all have degree 2 is well labeled iff the labels of neighboring vertices differ by at most 1.

Chapuy–Dołęga bijection



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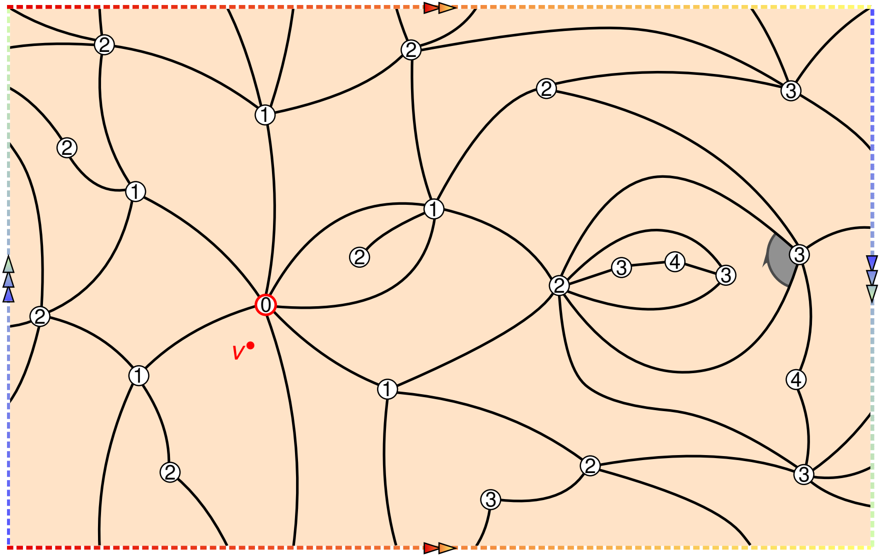

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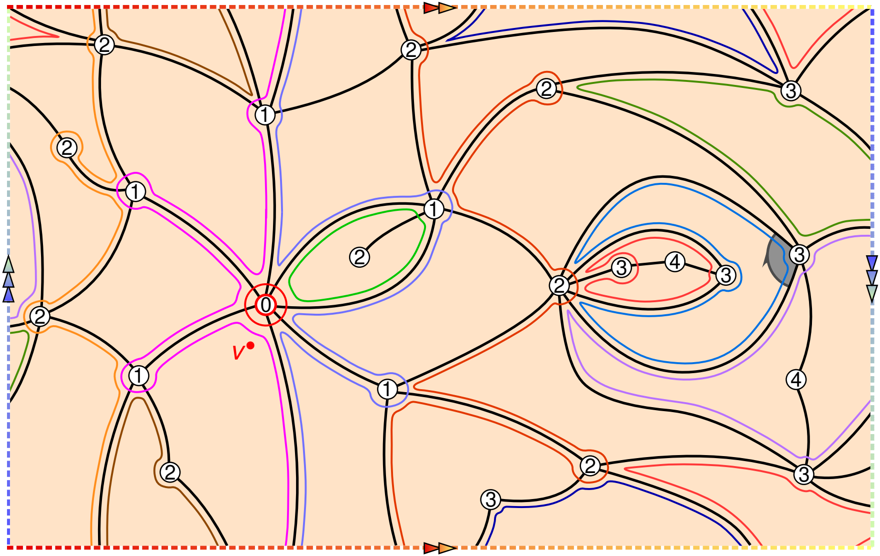

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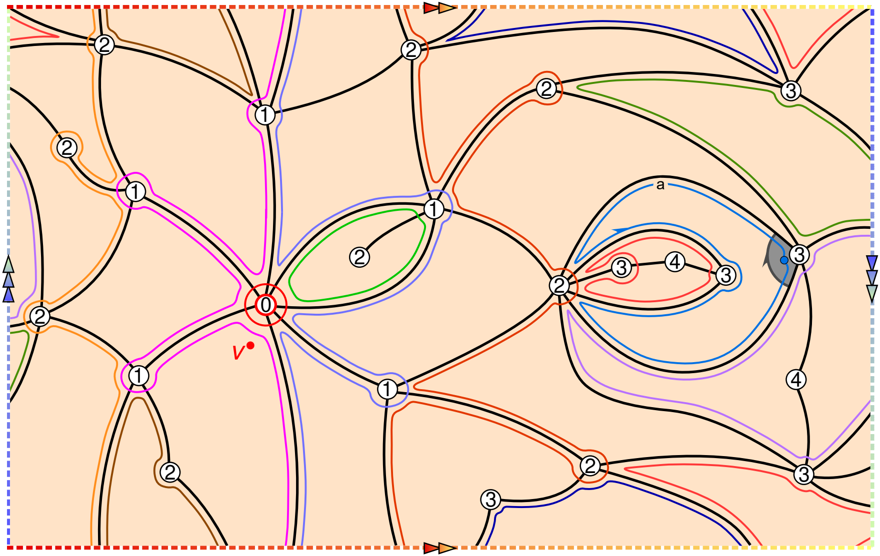

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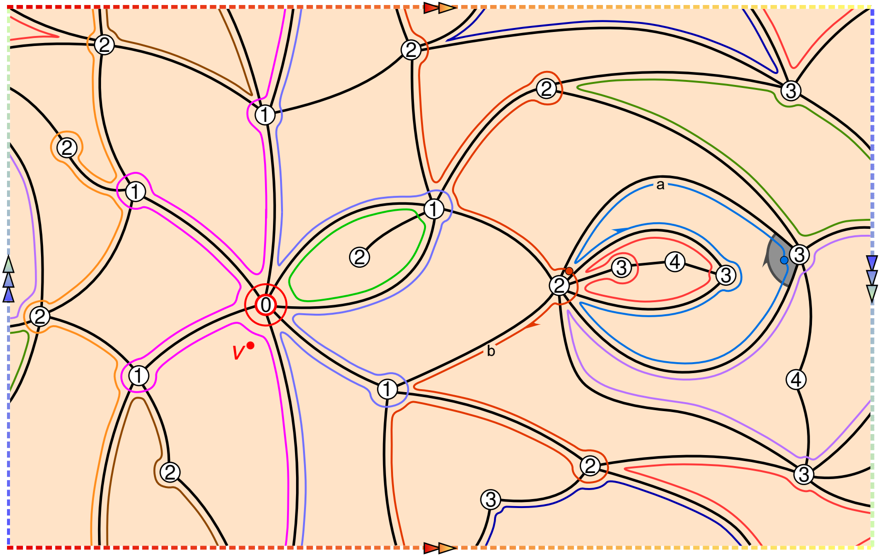

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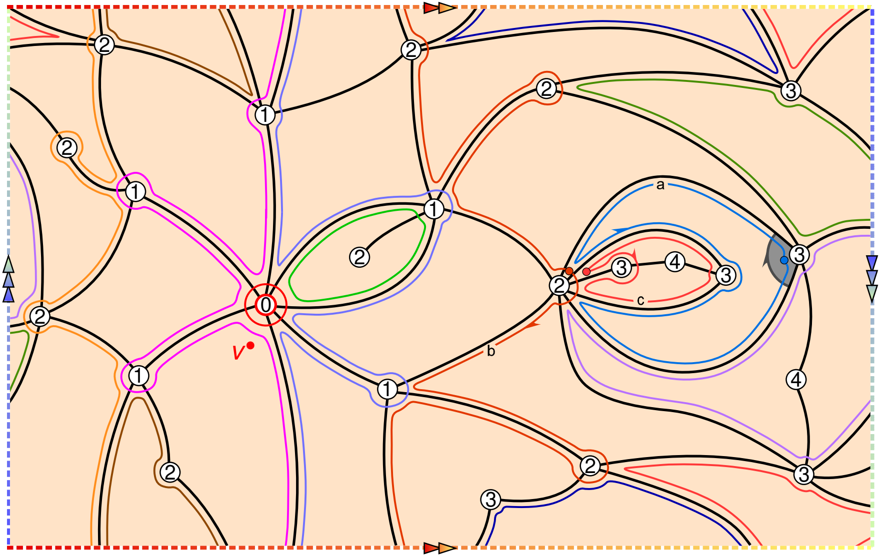

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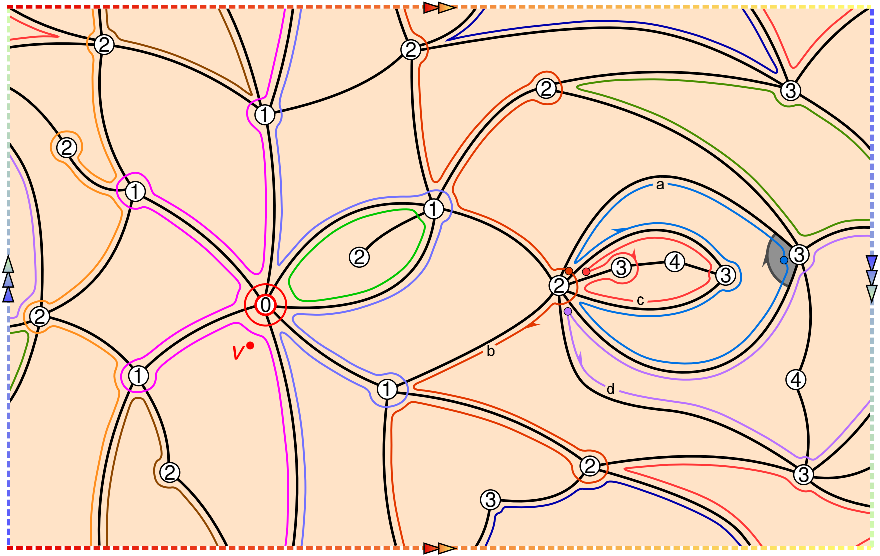

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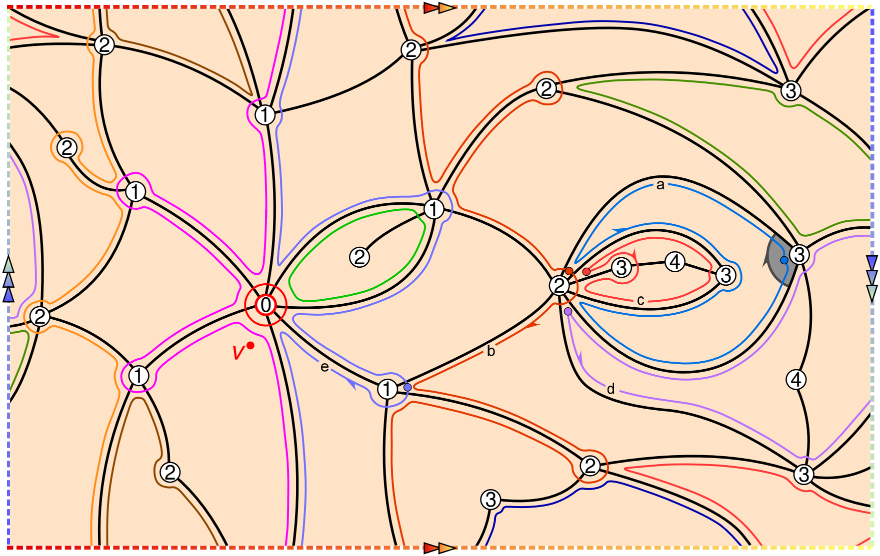

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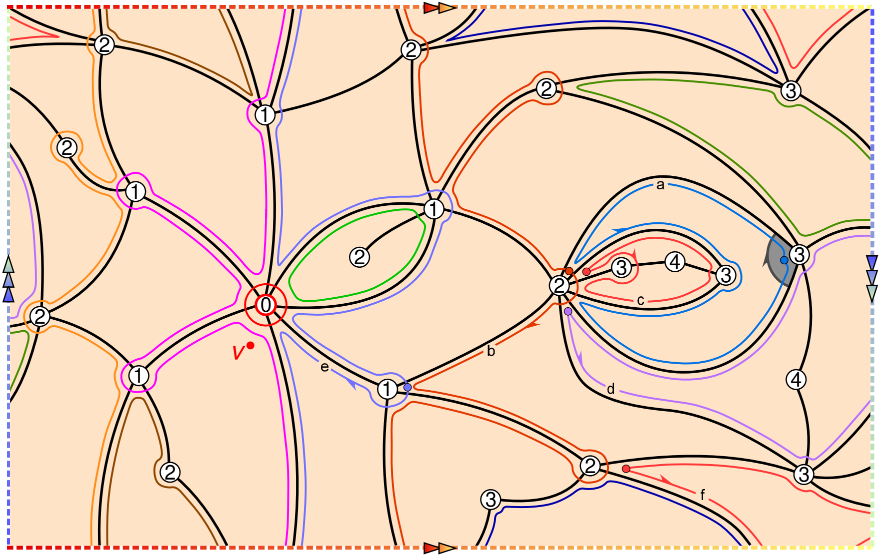

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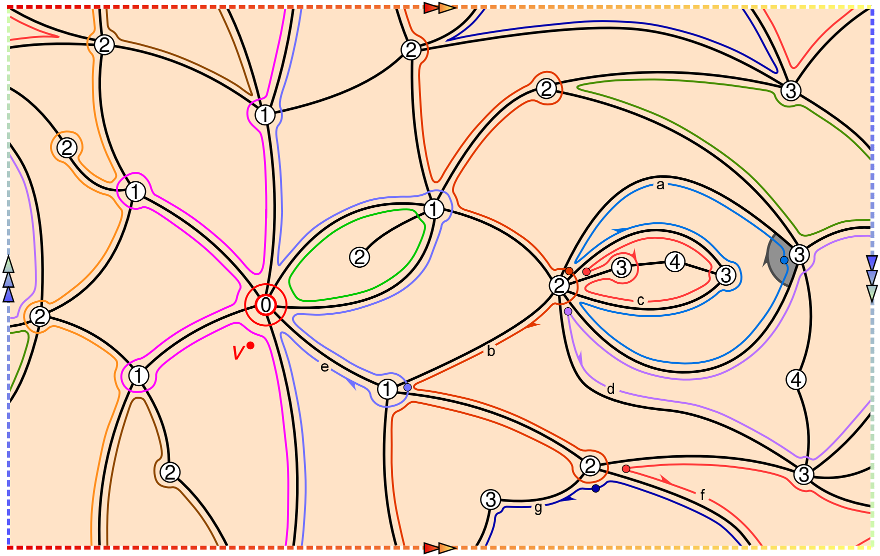

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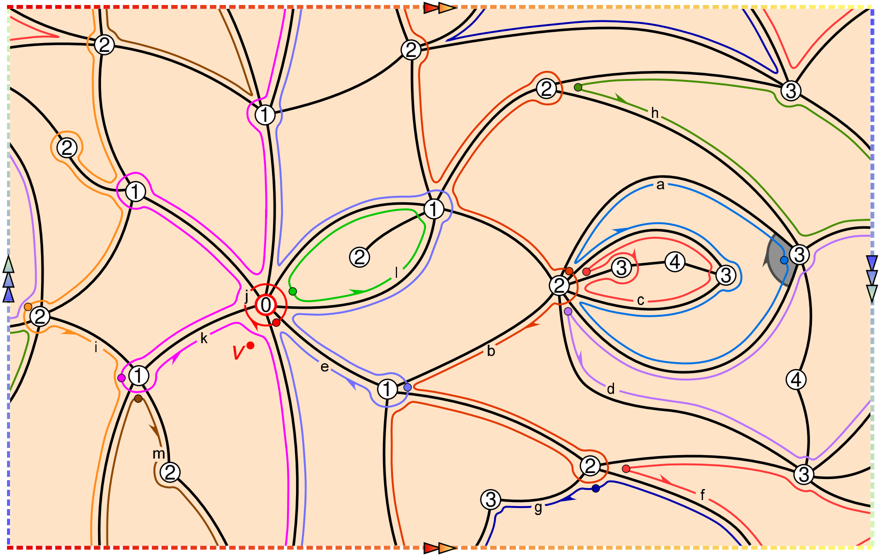

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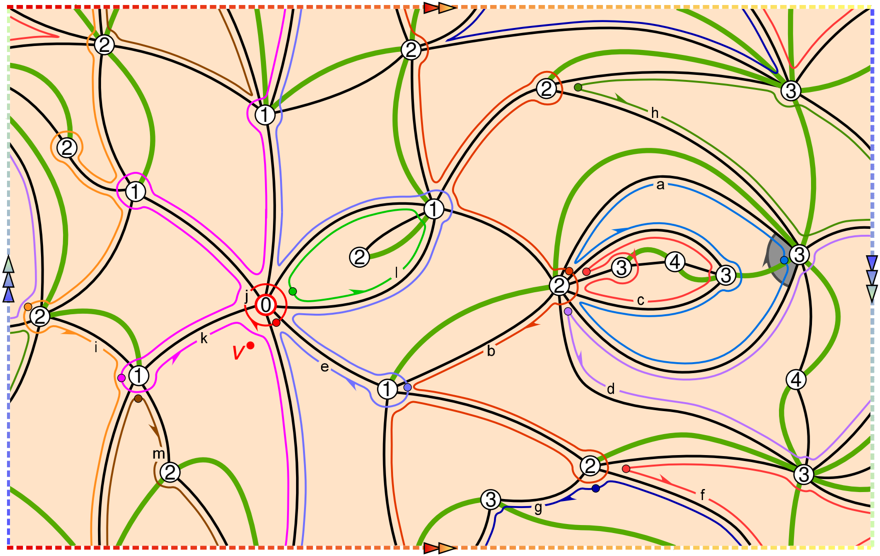

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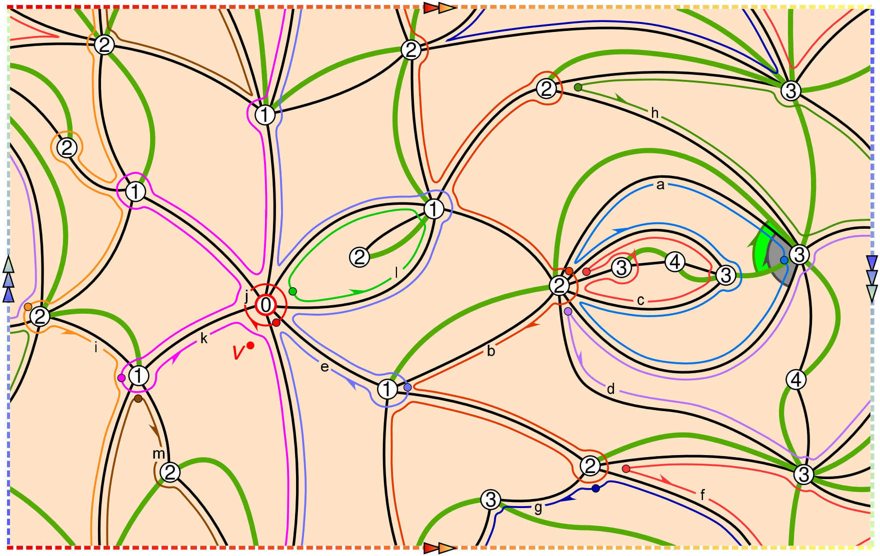

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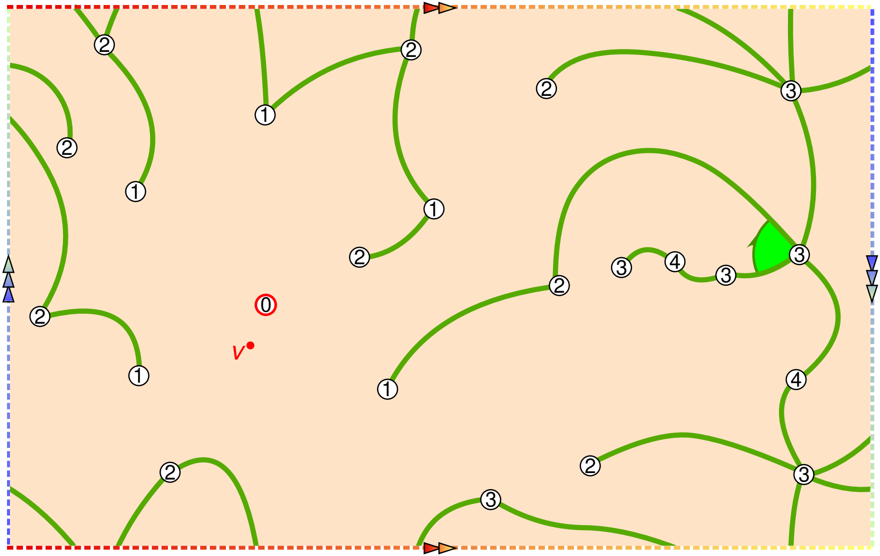

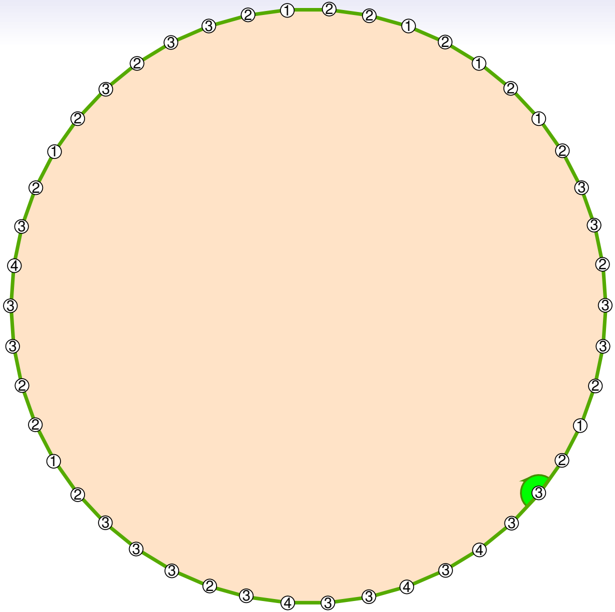
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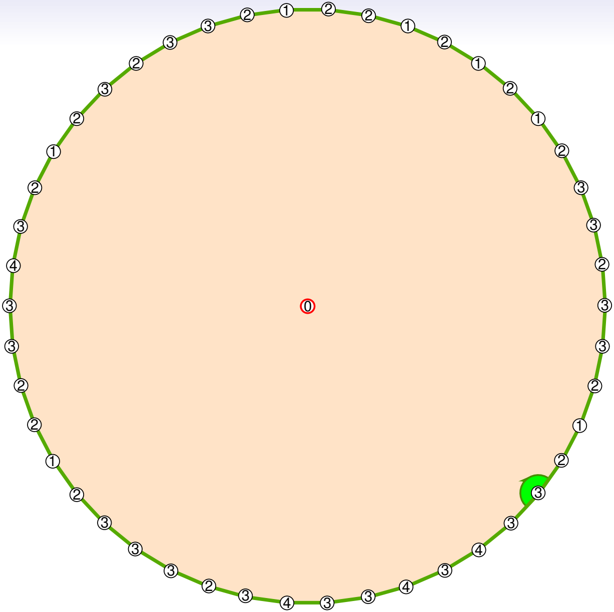
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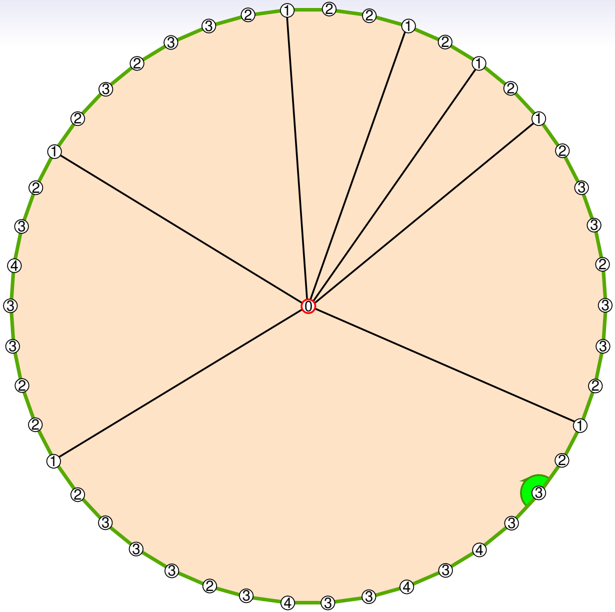
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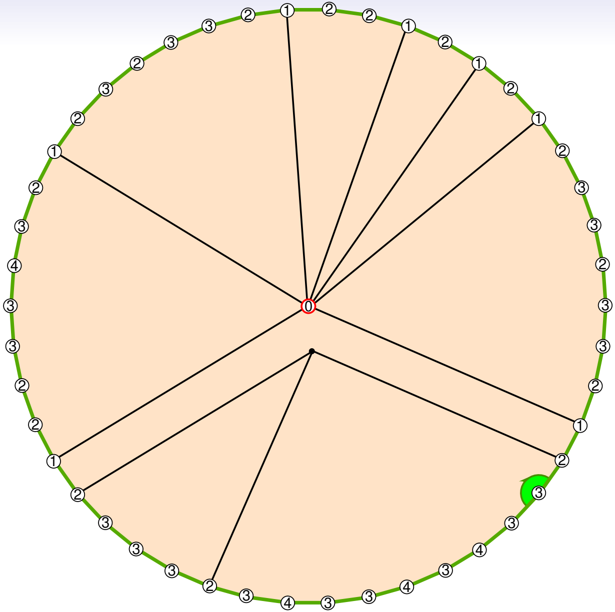

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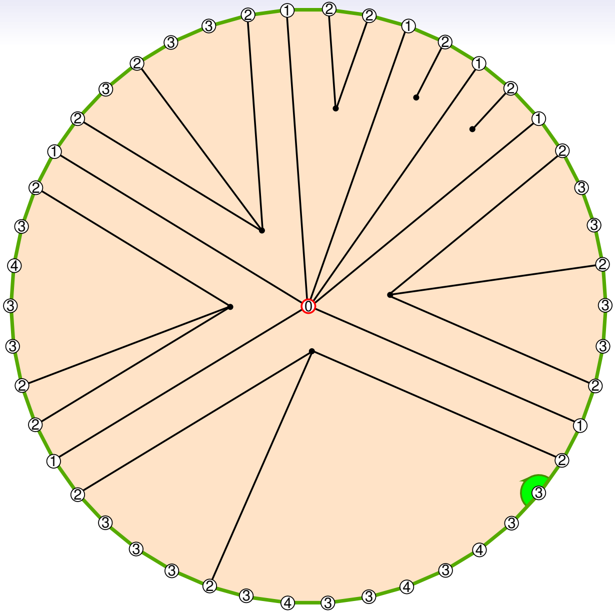
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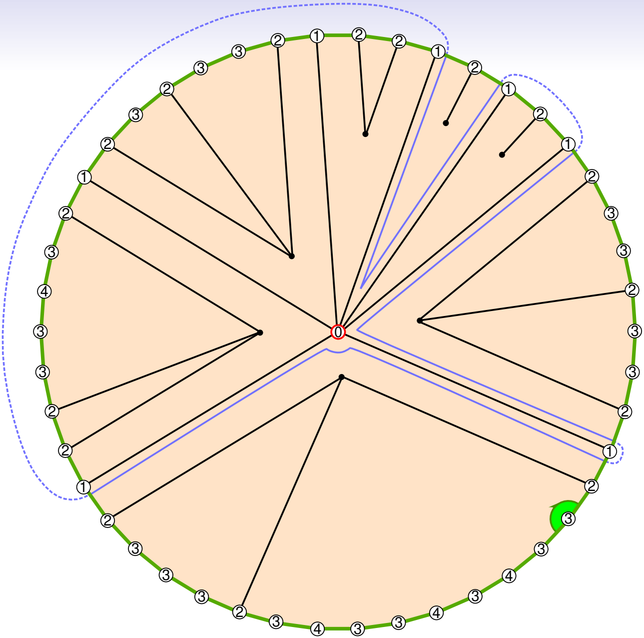

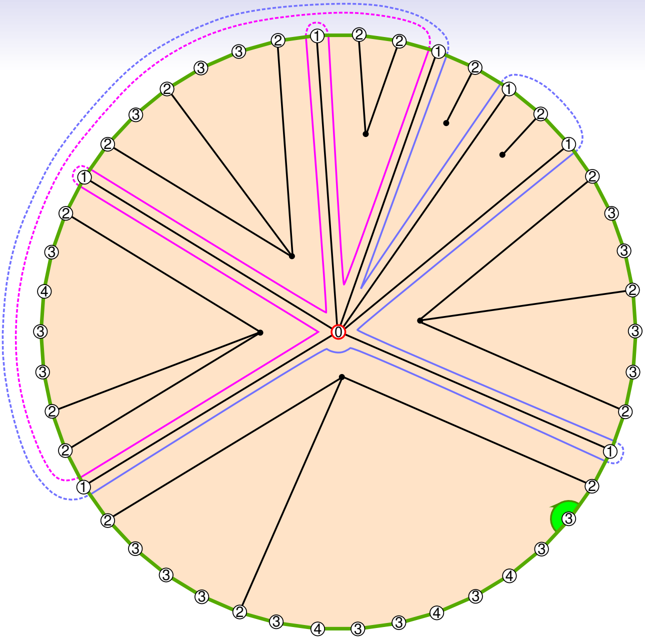
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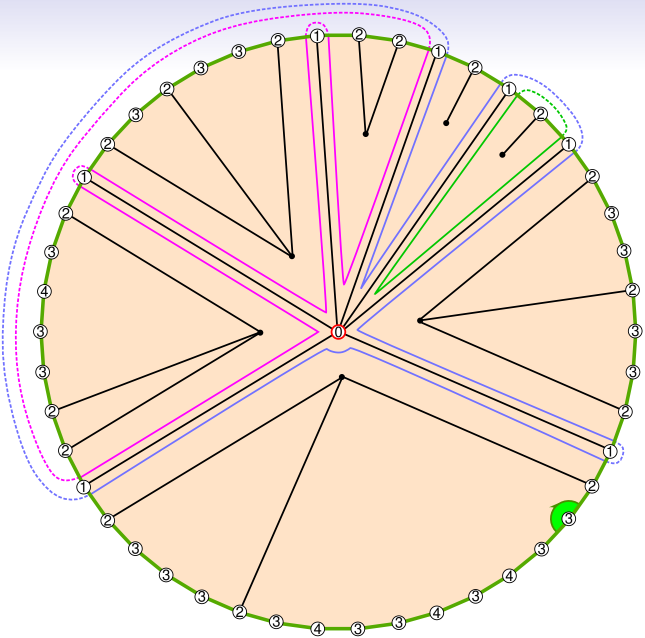

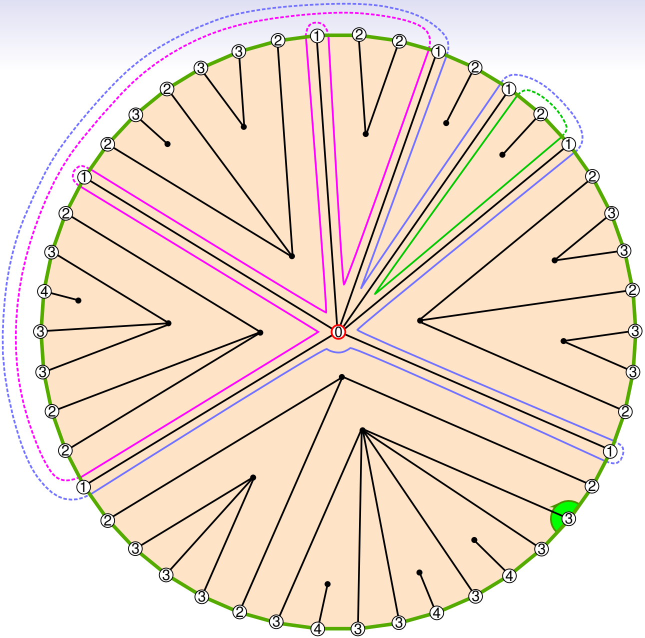

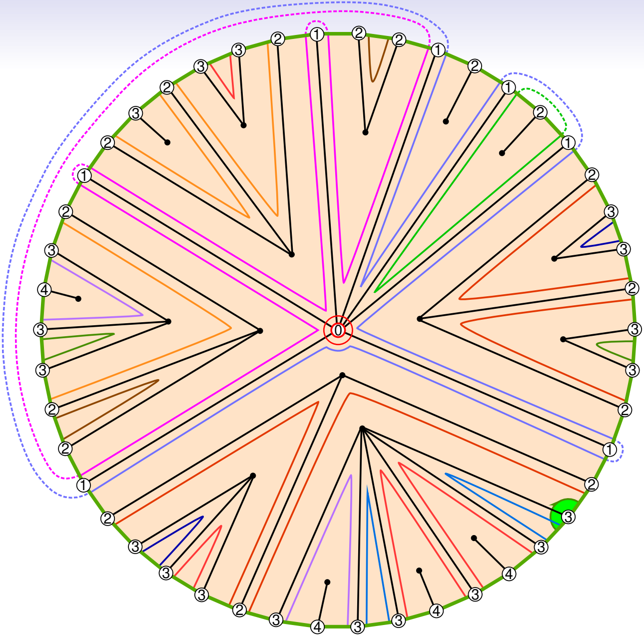

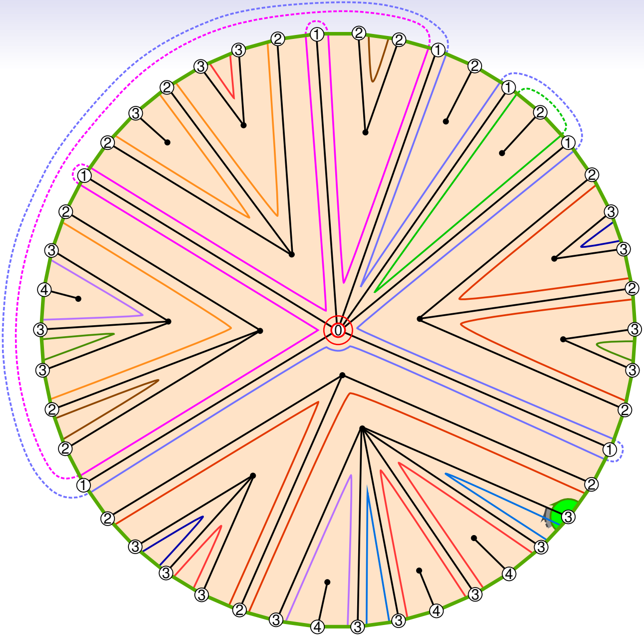

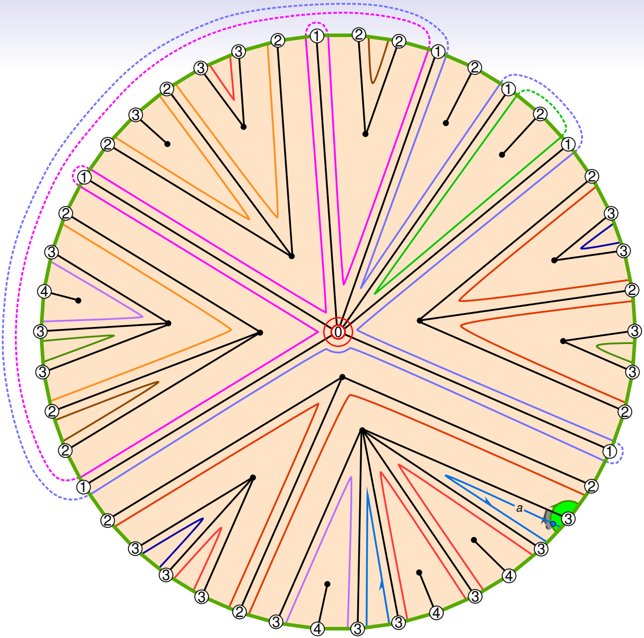
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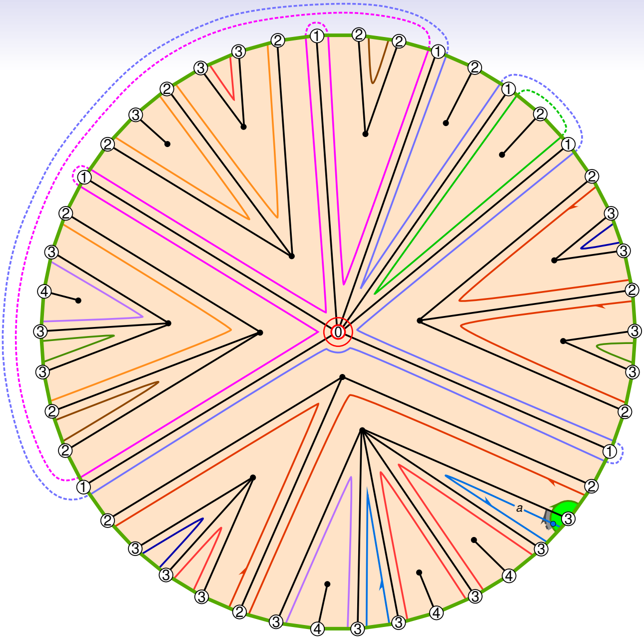

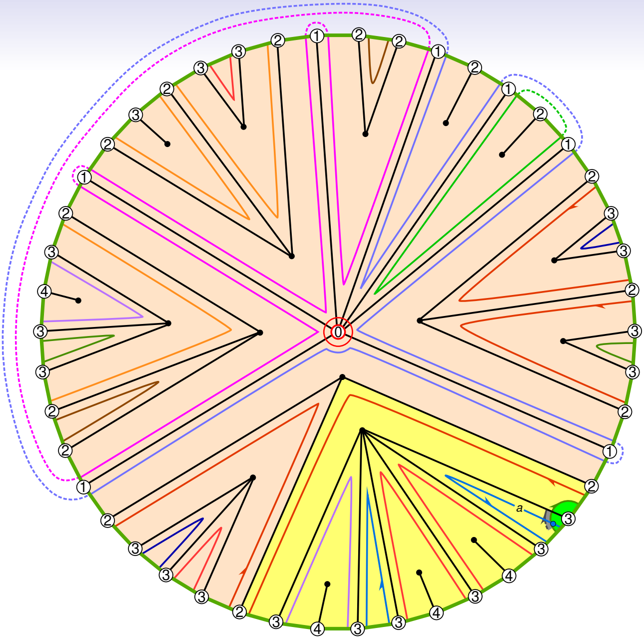

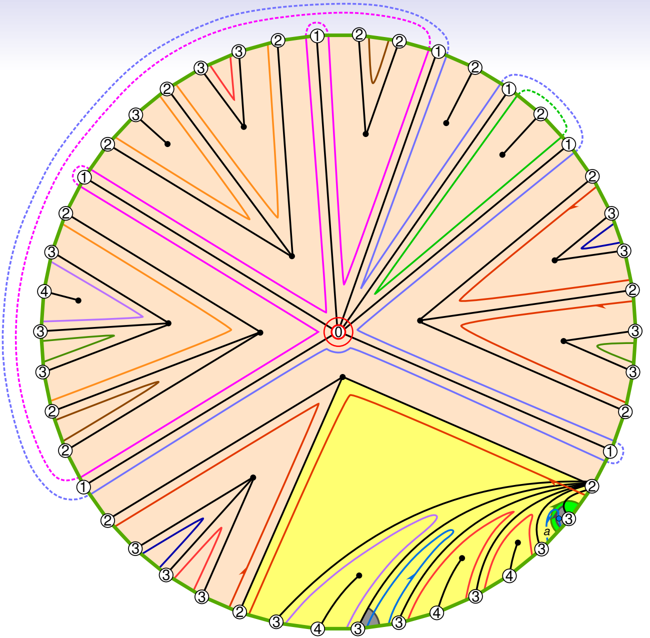

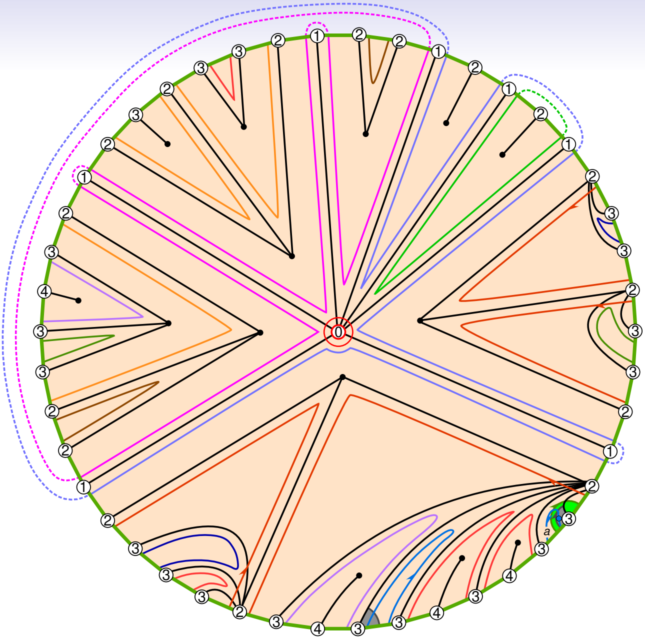

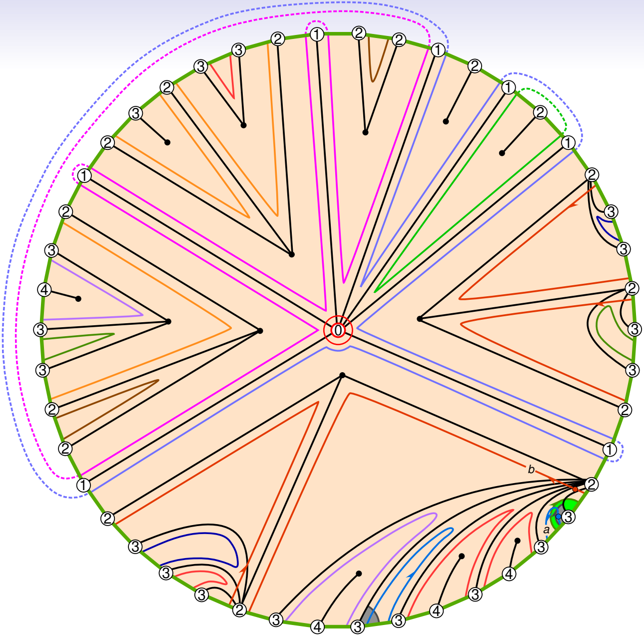

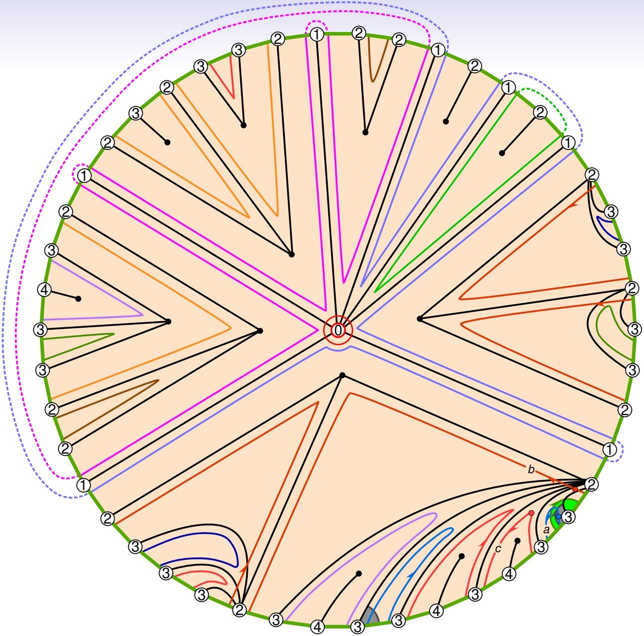

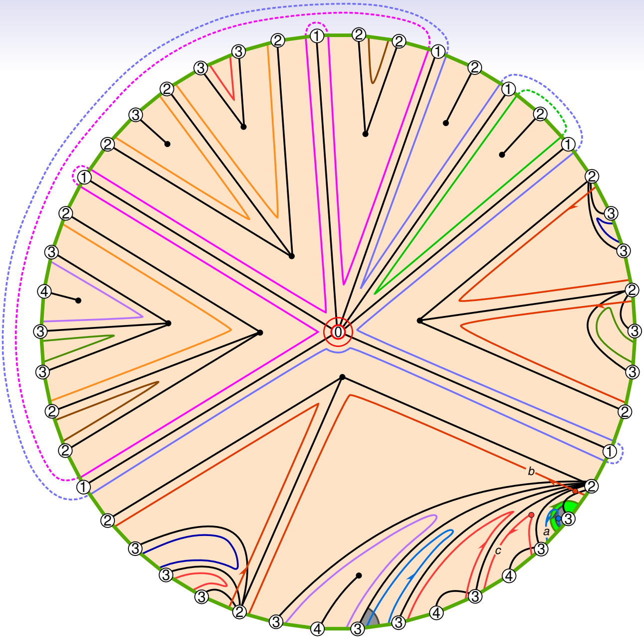

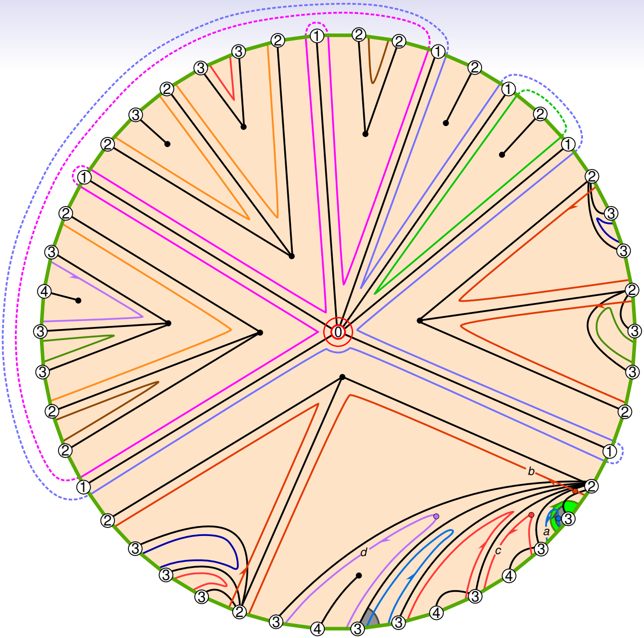
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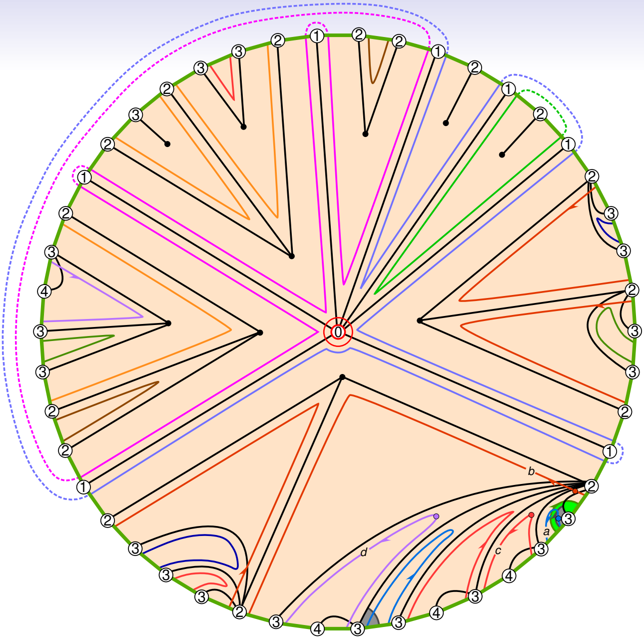
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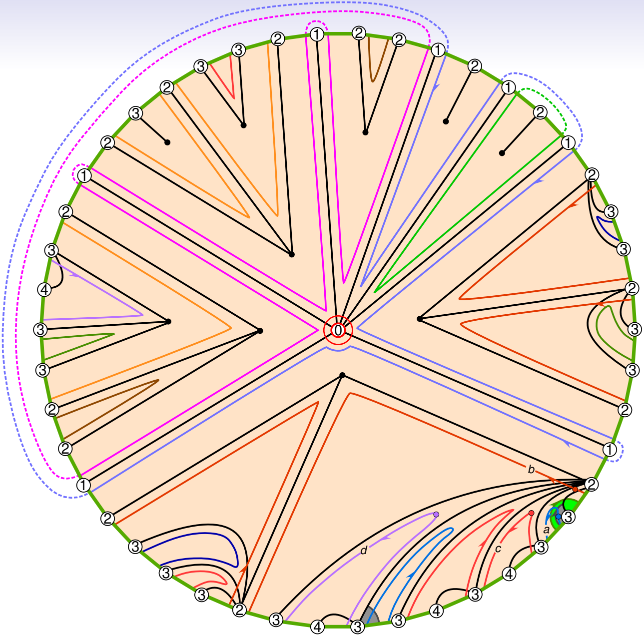
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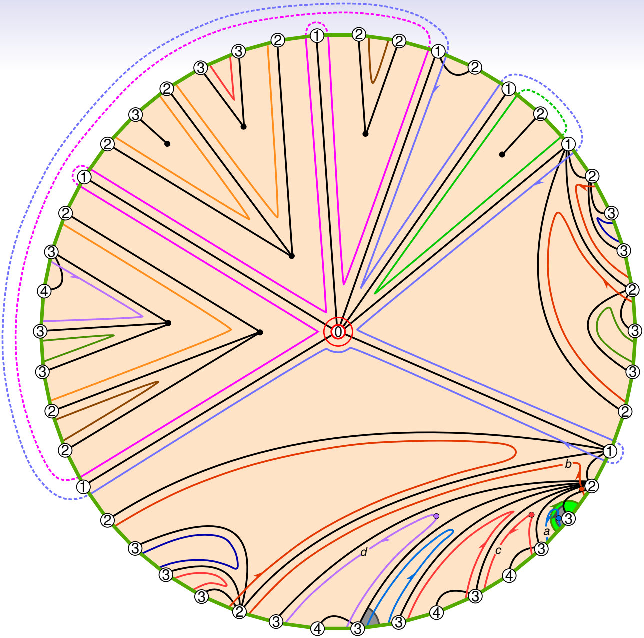
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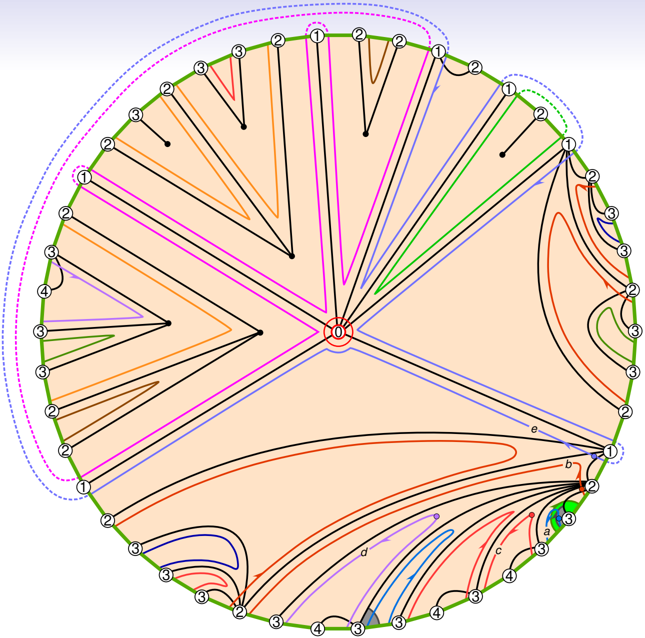

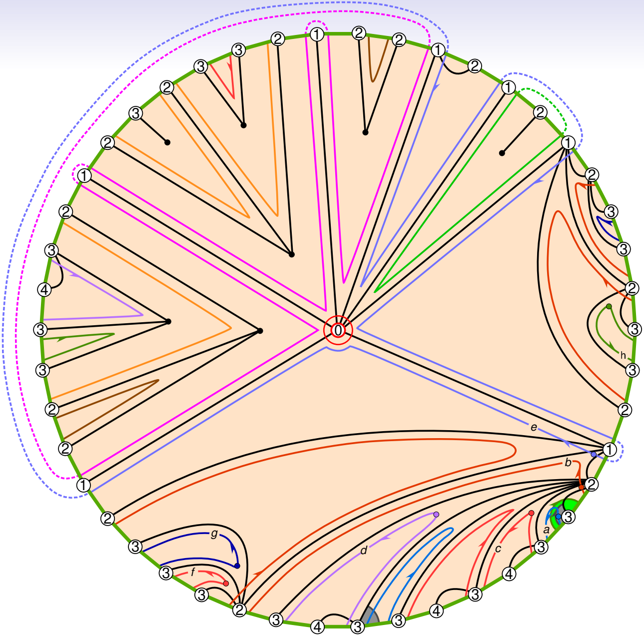
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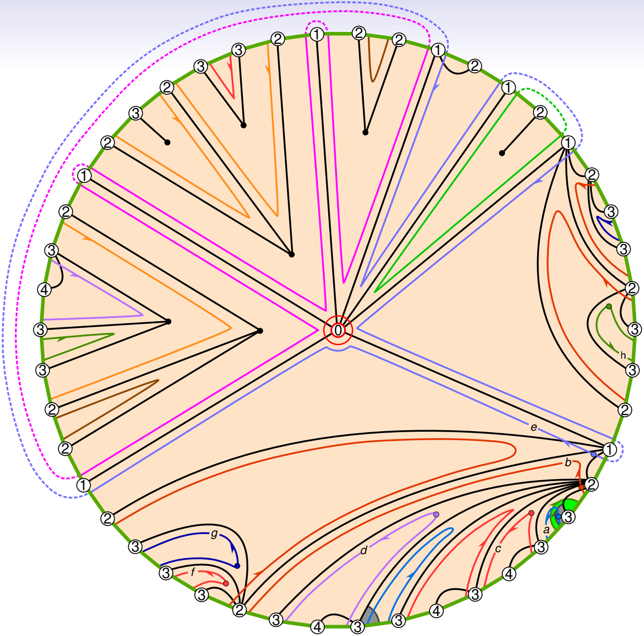
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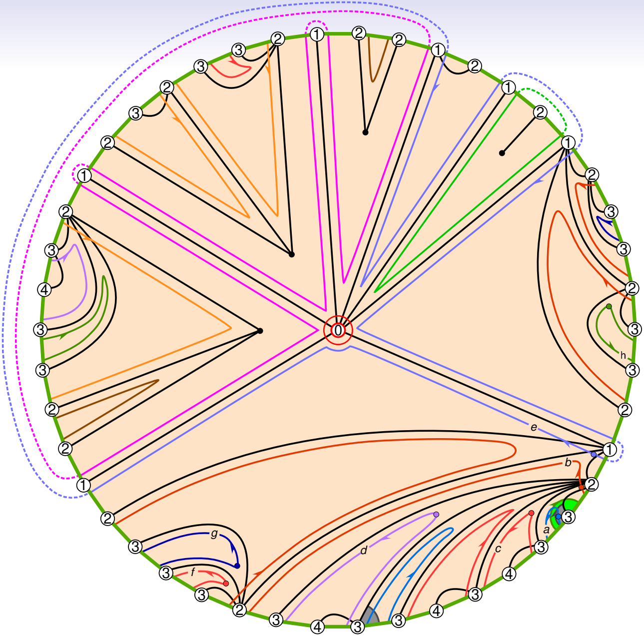
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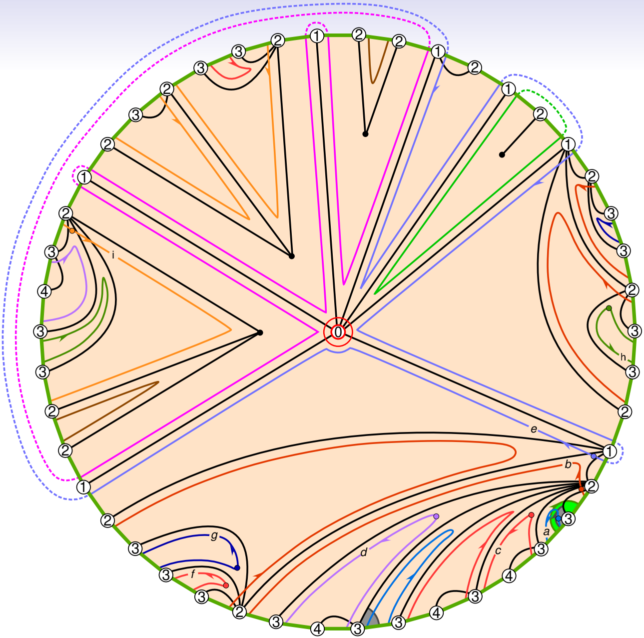
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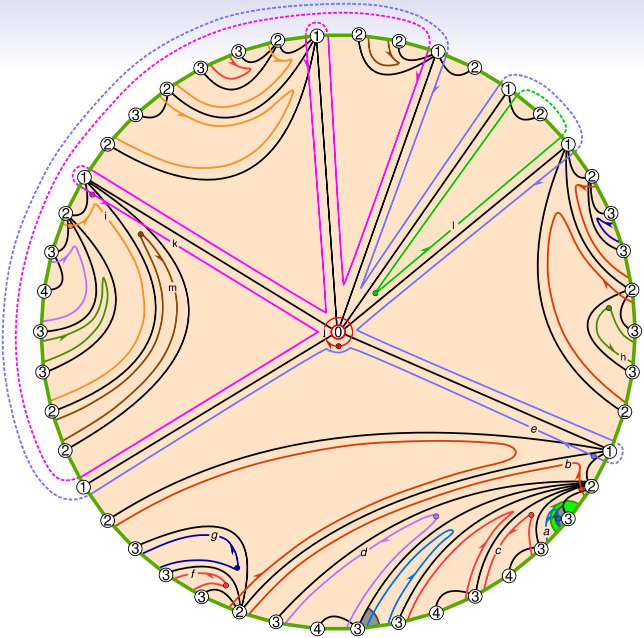
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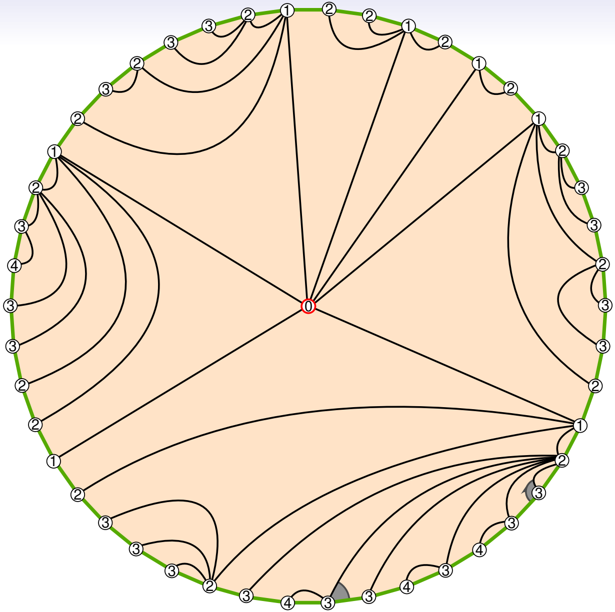

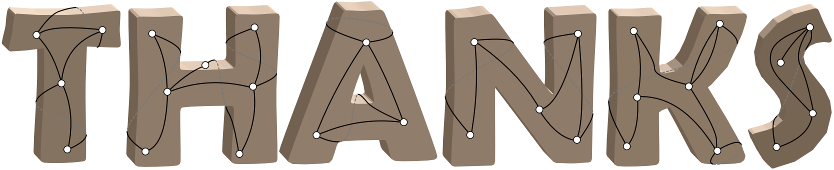
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