Convergence of uniform noncrossing partitions to the Brownian triangulation

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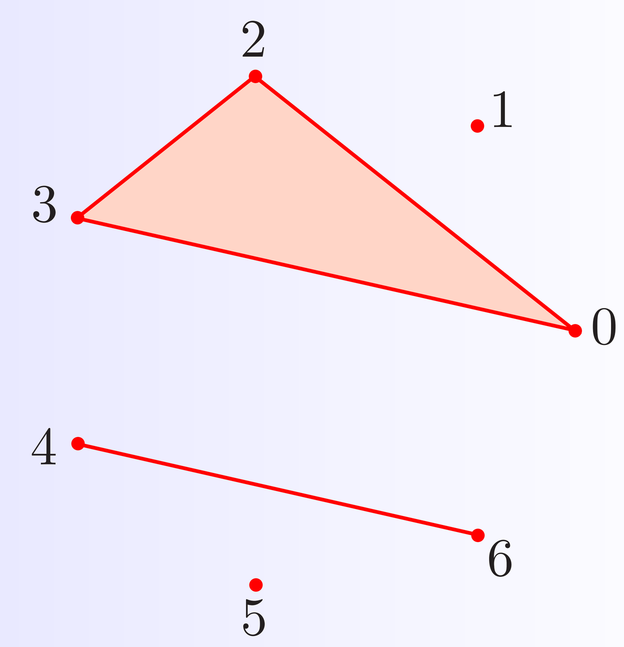
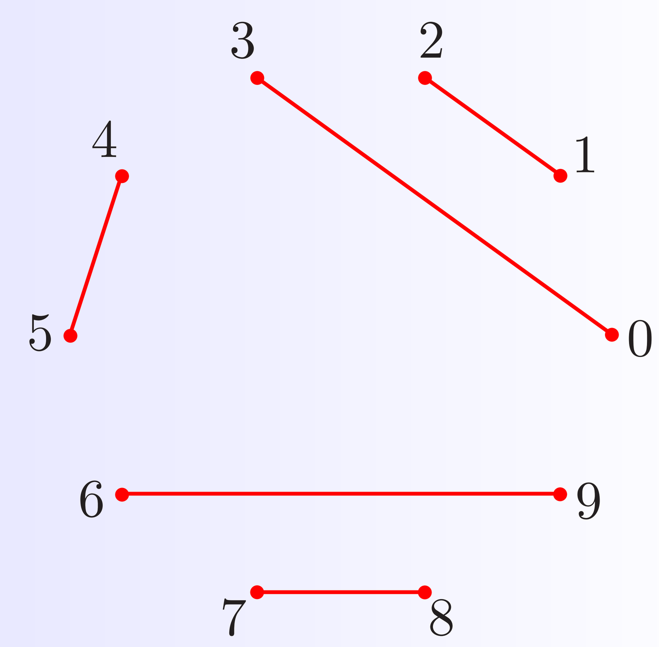
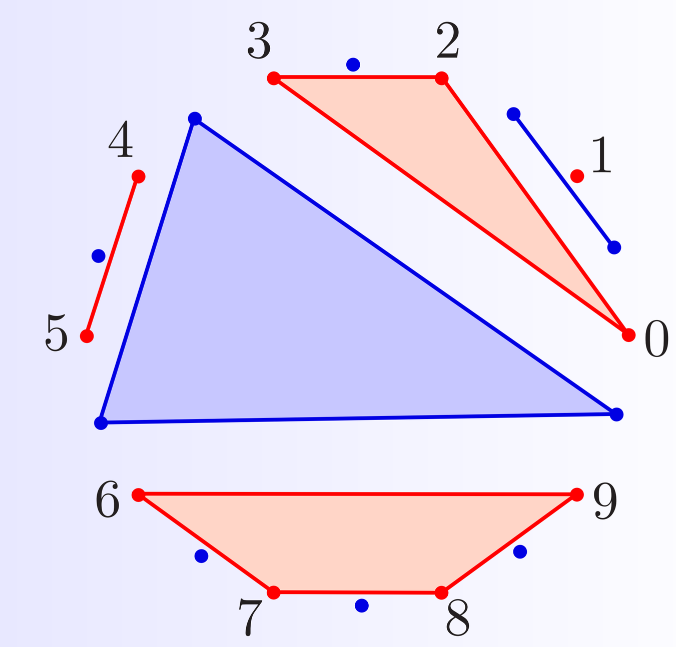
Noncrossing partition



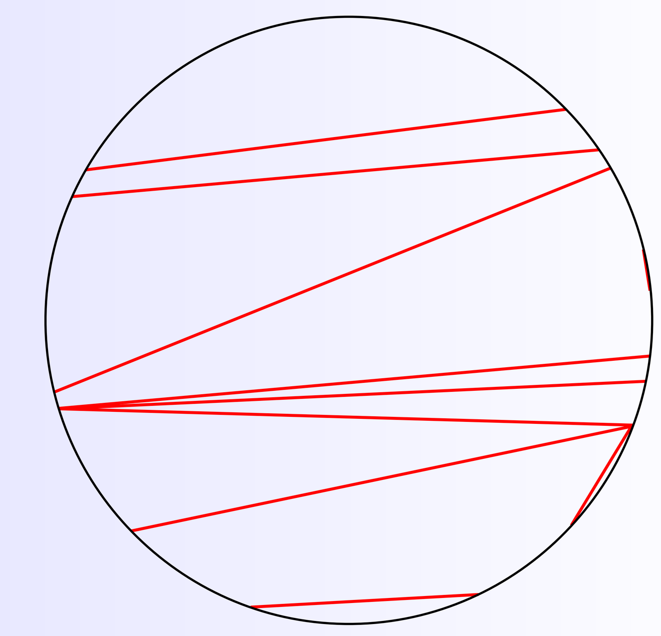
Noncrossing pair part.



Kreweras complement

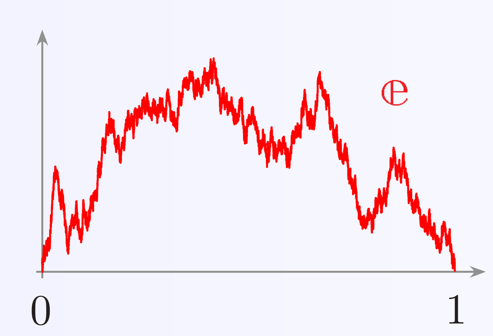


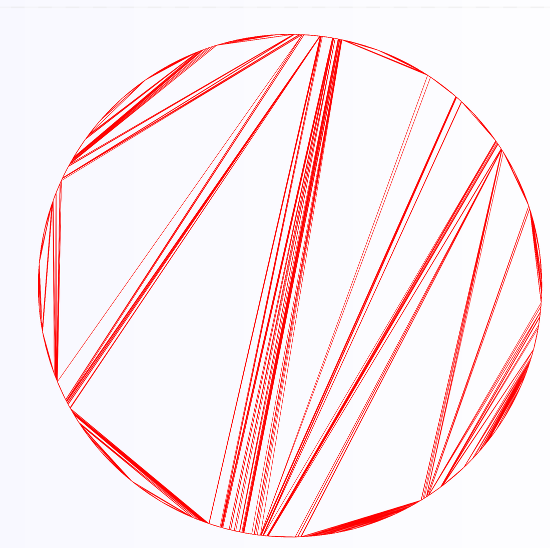
Lamination



The Brownian triangulation $\mathcal{B}$ [Aldous]

- Normalized Brownian excursion $(\mathbb{e}_t)_{0\le t\le 1}$.
- Declare $s \overset{\mathbb{e}}{\sim} t$ when $\mathbb{e}_s = \mathbb{e}_t = \min_{s\wedge t\le r\le s\vee t} \mathbb{e}_r$.
- The *Brownian triangulation* is $\mathcal{B} := \bigcup_{s\overset{\mathbb{e}}{\sim} t} \left[e^{2i\pi s}, e^{2i\pi t}\right]$.

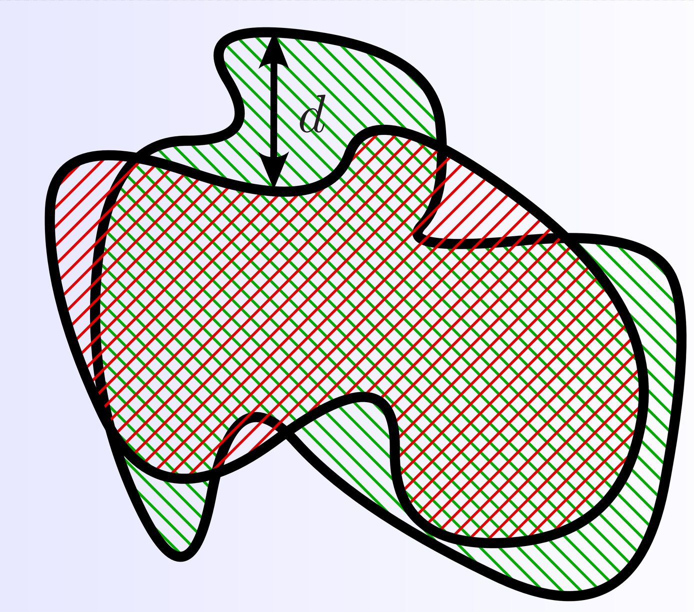




[Igor Kortchemski]

Prop. *A.s., $\mathcal{B}$ is compact and each bounded connected component of its complement is an open Euclidean triangle whose vertices belong to the unit circle.*

Hausdorff distance

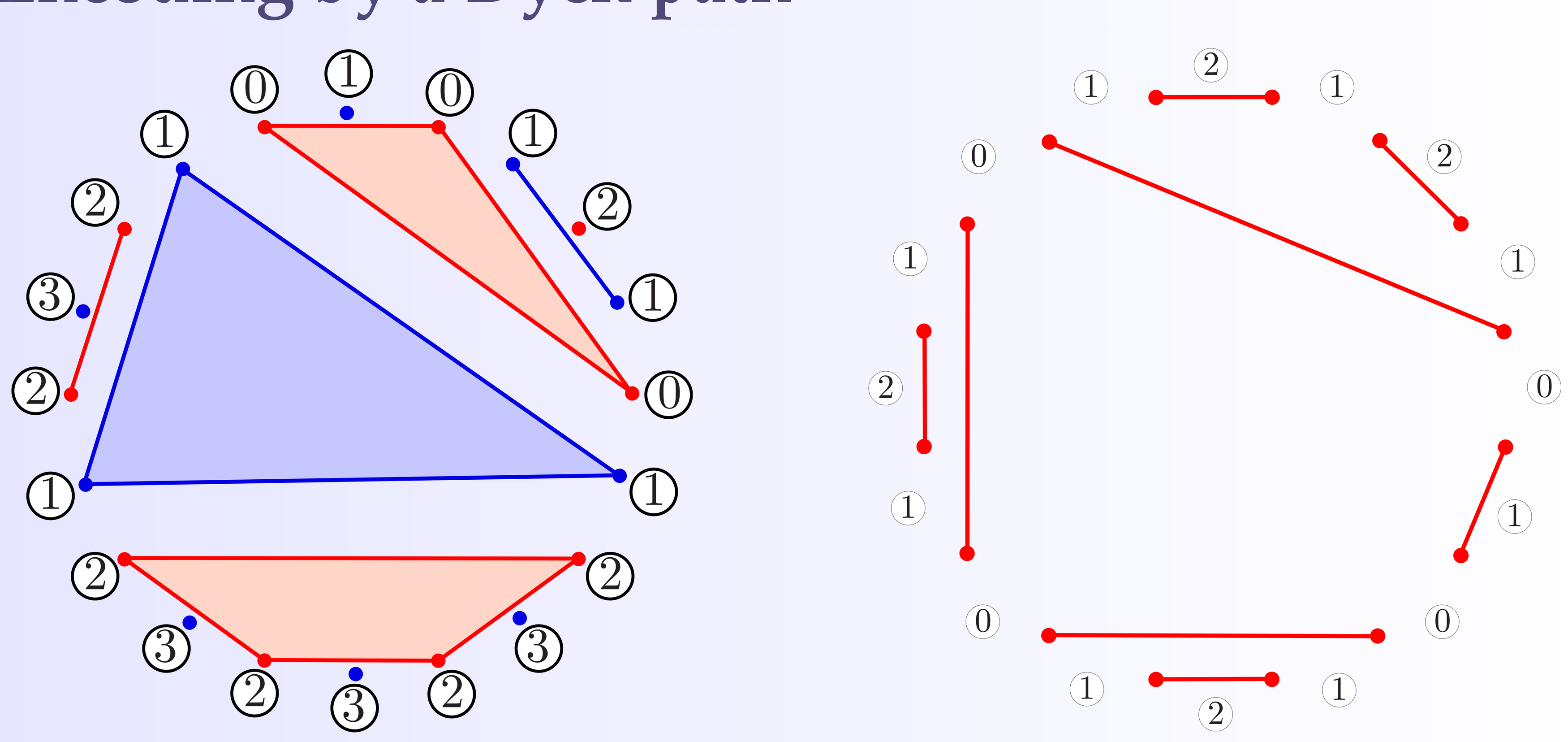


Convergence toward the Brownian triangulation

Thm (Curien–Kortchemski '14, B. '17). *Seen as a lamination, a uniform noncrossing partition of size n tends in law to $\mathcal{B}$, for the Hausdorff topology.*

Thm (Curien–Kortchemski '14, B. '17). *As a lamination, a uniform noncrossing **pair** partition of size $2n$ tends in law to $\mathcal{B}$, for the Hausdorff topology.*

Encoding by a Dyck path



Proof

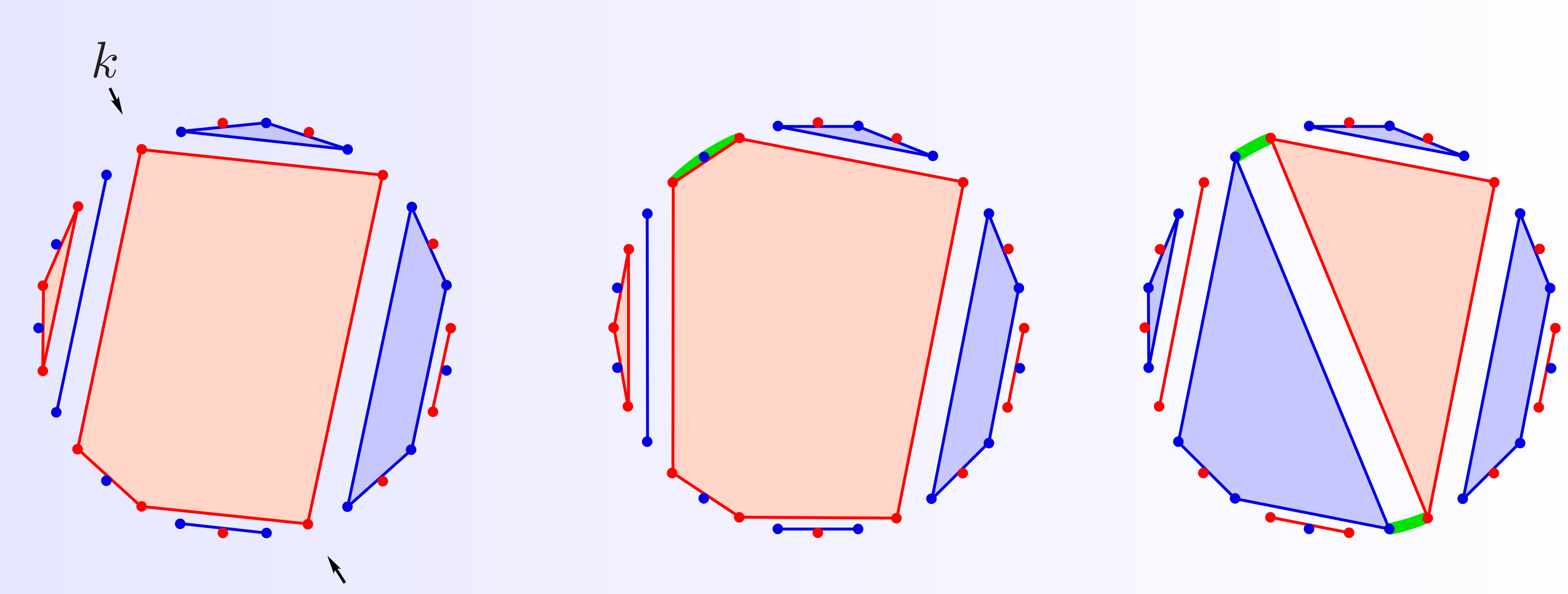
- Let $\mathcal{P}_n$ be a uniform noncrossing partition of size n.
- Let $L_n : [0,1] \to \mathbb{R}_+$ be the encoding Dyck path ($L_n(k/2n)$: label of the k-th $2n$-th root of unity, and linear interpolation).
- Conditioned version of Donsker: $\left(\frac{L_n(s)}{\sqrt{2n}}\right)_{0\le s\le 1} \xrightarrow{(d)} (\mathbb{e}_s)_{0\le s\le 1}$.
- *(Assume a.s. convergence by Skorokhod's theorem.)*
- By compactness, $(\mathcal{P}_n)_n$ has accumulation points. Let $\mathcal{P}$ be one.
- As the local minimums of $\mathbb{e}$ on $(0,1)$ are distinct, if $s \overset{\mathbb{e}}{\sim} t$ with $s<t$, we can find even $s_n, t_n \in \{0,2,4,\dots,2n\}$ such that $s_n < t_n$,

$$\frac{s_n}{2n} \to s\,, \quad \frac{t_n}{2n} \to t \quad \text{and} \quad L_n\Big(\frac{s_n}{2n}\Big) = L_n\Big(\frac{t_n}{2n}\Big) < \min_{\left[\frac{s_n+1}{2n}, \frac{t_n-1}{2n}\right]} L_n\,.$$

- The chord $\left[e^{i\frac{2\pi}{2n}s_n}, e^{i\frac{2\pi}{2n}t_n}\right] \subseteq \mathcal{P}_n$, so that $\left[e^{2i\pi s}, e^{2i\pi t}\right] \subseteq \mathcal{P}$. Thus $\mathcal{B} \subseteq \mathcal{P}$.
- $\mathcal{B}$ is maximal for the inclusion relation, so that $\mathcal{B} = \mathcal{P}$. ☺

Growing algorithm for noncrossing partitions

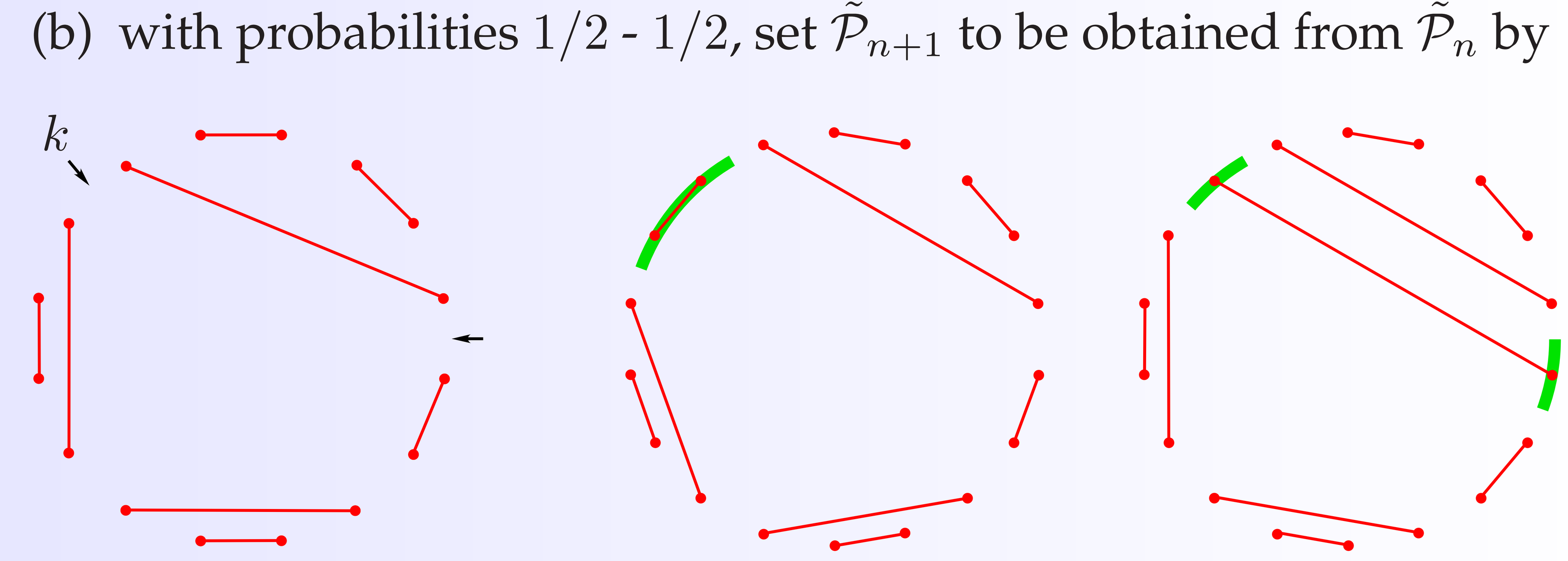
1. Let $\mathcal{P}_1$ be the only partition of size 1.
2. Generate $\mathcal{P}_{n+1}$ from $\mathcal{P}_n$ as follows:
   (a) choose an integer k uniformly at random in $\{0,1,\dots,2n\}$;
   (b) with probabilities 1/2 - 1/2, set $\mathcal{P}_{n+1}$ to be obtained from $\mathcal{P}_n$ by



Prop. *$\mathcal{P}_n$ is a uniform noncrossing partition of size n. Moreover, seen as a lamination, $\mathcal{P}_n$ almost surely converges toward $\mathcal{B}$, for the Hausdorff topology.*

Growing algorithm for noncross. pair partitions

1. Let $\tilde{\mathcal{P}}_1$ be the only pair partition of size 2.
2. Generate $\tilde{\mathcal{P}}_{n+1}$ from $\tilde{\mathcal{P}}_n$ as follows:
   (a) choose an integer k uniformly at random in $\{0,1,\dots,2n\}$;
   (b) with probabilities 1/2 - 1/2, set $\tilde{\mathcal{P}}_{n+1}$ to be obtained from $\tilde{\mathcal{P}}_n$ by



Prop. *$\tilde{\mathcal{P}}_n$ is a uniform noncrossing pair partition of size $2n$. Moreover, seen as a lamination, $\tilde{\mathcal{P}}_n$ almost surely converges to $\mathcal{B}$, for the Hausdorff topology.*