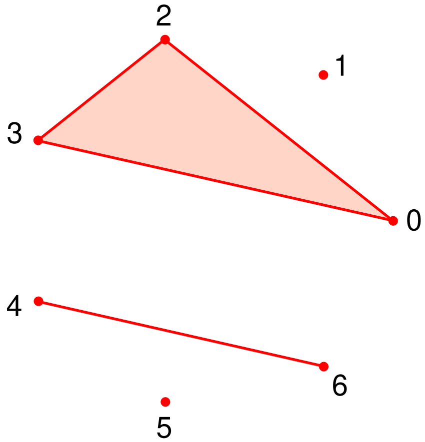
Convergence of uniform noncrossing partitions toward the Brownian triangulation

Jérémie BETTINELLI

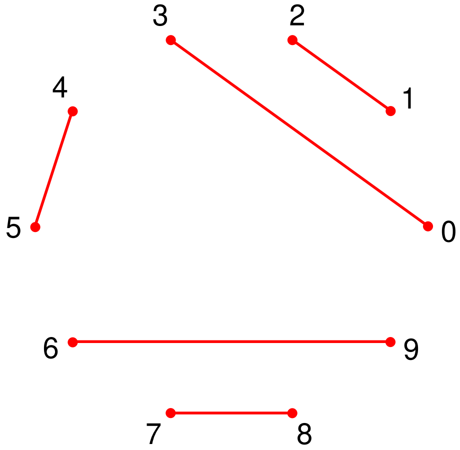
January 17, 2019

Noncrossing partition



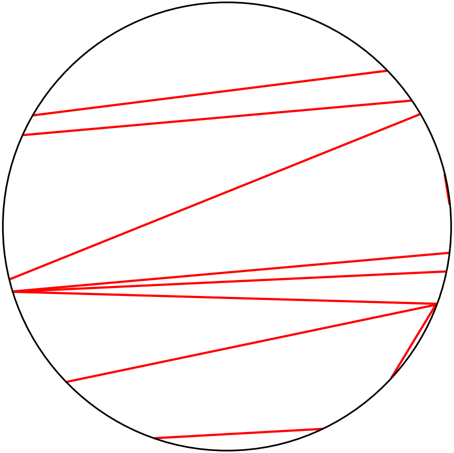
Noncrossing partition: partition of the n-th roots of unity for some $n \geq 1$, such that the convex hulls of its blocks are pairwise disjoint

Noncrossing pair partition



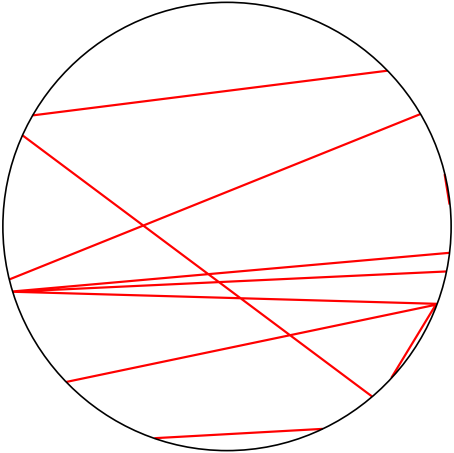
Noncrossing pair partition: noncrossing partition whose blocks are all of size exactly 2

Lamination



Lamination: closed subset of the unit disk $\overline{\mathbb{D}}$ consisting of a union of chords whose intersections with the open unit disk are pairwise disjoint

Lamination



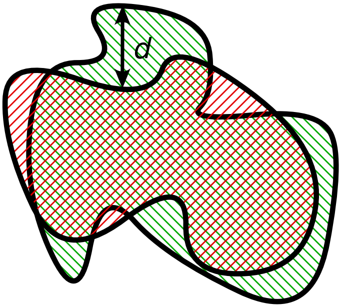
Lamination: closed subset of the unit disk $\overline{\mathbb{D}}$ consisting of a union of chords whose intersections with the open unit disk are pairwise disjoint

Hausdorff metric

The Hausdorff distance between closed subsets $A, B \subseteq \overline{\mathbb{D}}$ is

$$\inf \left\{ \varepsilon > 0 \;:\; A \subseteq B^{(\varepsilon)} \text{ and } B \subseteq A^{(\varepsilon)} \right\},$$

where $X^{(\varepsilon)} := \{ z \in \overline{\mathbb{D}} \;:\; d(z, X) \leq \varepsilon \}$.

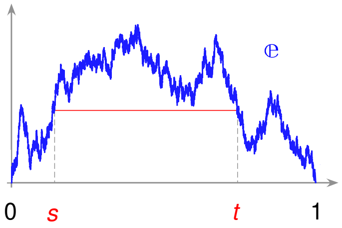


Endowed with the Hausdorff metric, the set of all closed subsets of $\overline{\mathbb{D}}$ is a compact metric space.

The Brownian triangulation $\mathcal{B}$ [Aldous]

- Take a normalized Brownian excursion $(\mathbb{e}_t)_{0\le t\le 1}$.
- For $s, t \in [0,1]$, declare $s \overset{\mathbb{e}}{\sim} t$ when $\mathbb{e}_s = \mathbb{e}_t = \min_{s\wedge t\le r\le s\vee t} \mathbb{e}_r$.
- The Brownian triangulation is

$$\mathcal{B} := \bigcup_{s \overset{\mathbb{e}}{\sim} t} \left[e^{2i\pi s}, e^{2i\pi t}\right].$$



A.s., $\mathcal{B}$ is a closed subset of $\overline{\mathbb{D}}$ and a continuous triangulation of $\overline{\mathbb{D}}$, that is, each connected component of $\overline{\mathbb{D}} \setminus \mathcal{B}$ is an open Euclidean triangle whose vertices belong to the unit circle.

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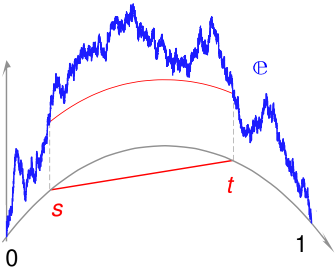

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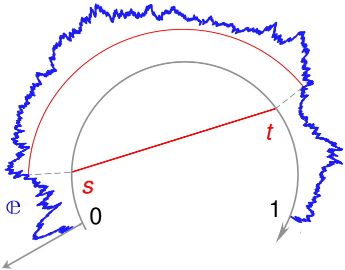

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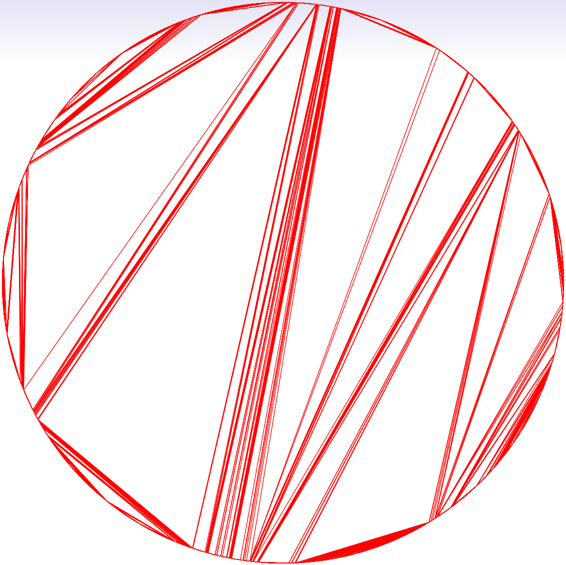

[simulation by Igor Kortchemski]

The theorems

Theorem (Curien–Kortchemski '14, B. '17)

Let $\mathcal{P}_n$ be a uniform noncrossing partition of size n, seen as a lamination. Then $\mathcal{P}_n \xrightarrow{(d)} \mathcal{B}$, for the Hausdorff topology.

Theorem (Curien–Kortchemski '14, B. '17)

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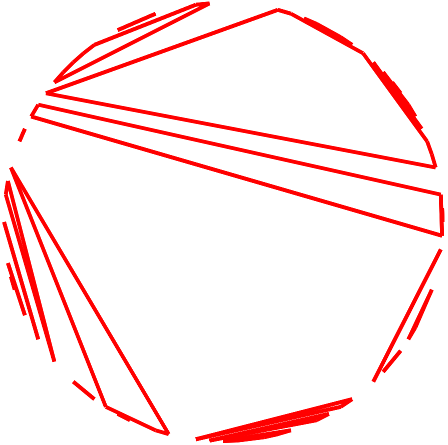
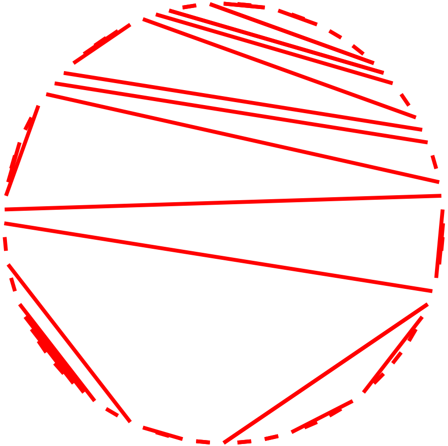
- setting proposed for uniform triangulations [Aldous '94]
- uniform dissections, non-crossing trees [Curien–Kortchemski '14]
- "stable" analogs [Kortchemski '12, Kortchemski–Marzouk '16]
- recursive triangulations [Curien–Le Gall '11]
- simply generated noncrossing part. [Kortchemski–Marzouk '17]

Uniformly sampled examples of size 100



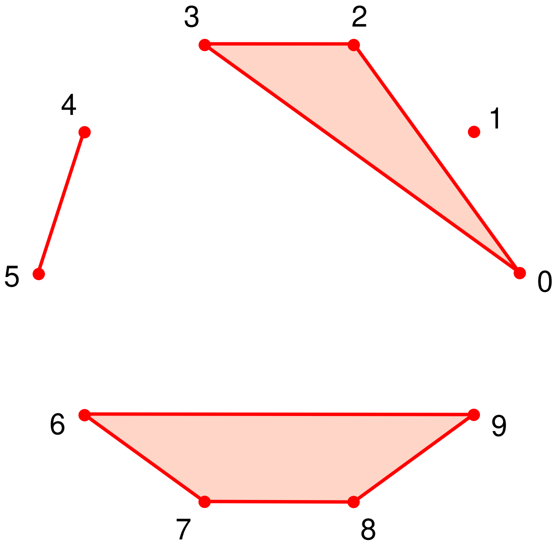
noncrossing partition



noncrossing pair partition

[simulations by Igor Kortchemski]

Kreweras complement



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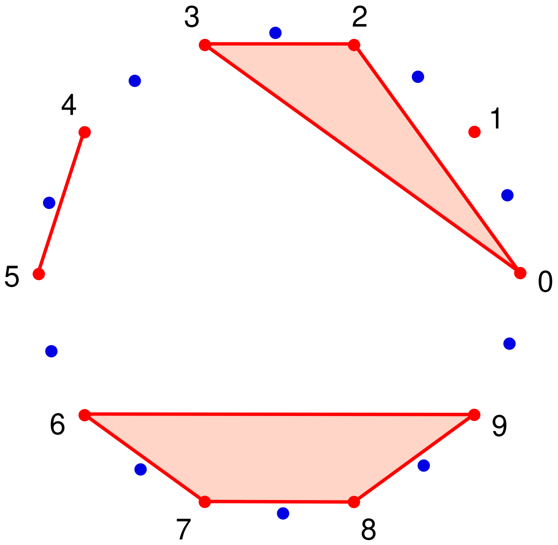

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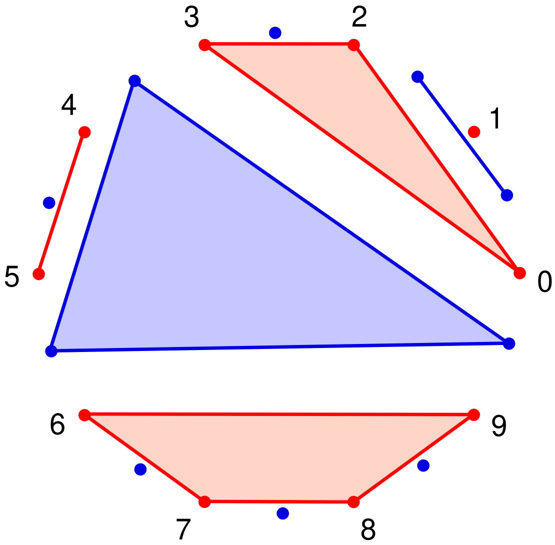
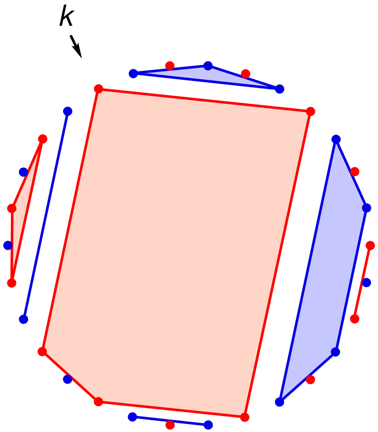
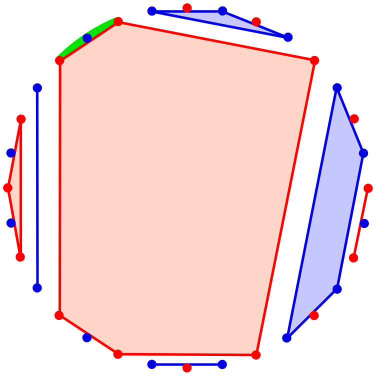


Operation 1: inserting a vertex at position k

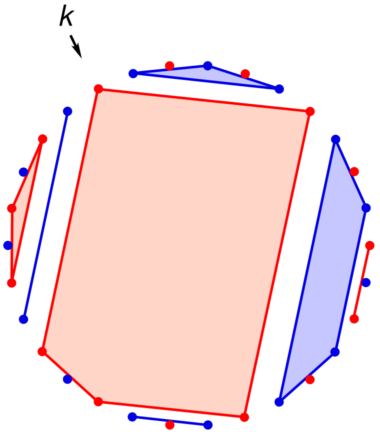
Data: a noncrossing partition of size n and an index $k \in \{0, 1, \ldots, 2n\}$

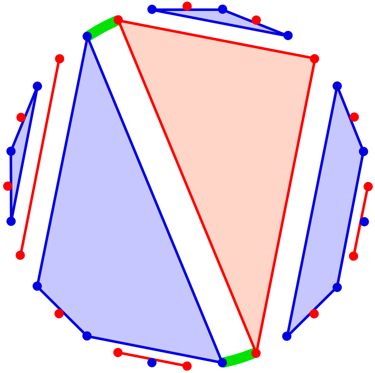




Operation 2: slicing at position k

Data: a noncrossing partition of size n and an index $k \in \{0, 1, \ldots, 2n\}$





Growing algorithm

Algorithm

1. *Let* $\mathcal{P}_1$ *be the only partition of size* 1*.*
2. *Generate* $\mathcal{P}_{n+1}$ *from* $\mathcal{P}_n$ *as follows:*
   1. *choose an integer* k *uniformly at random in* $\{0, 1, \ldots, 2n\}$*;*
   2. *with probabilities* 1/2 - 1/2*, set* $\mathcal{P}_{n+1}$ *to be obtained from* $\mathcal{P}_n$
      - *either by inserting a vertex at position* k*,*
      - *or by slicing at position* k*.*

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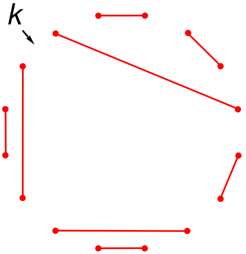
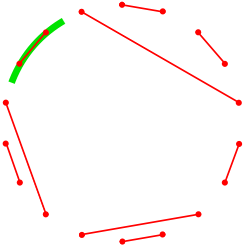
Proposition

$\mathcal{P}_n$ is a uniform noncrossing partition of size n. Moreover, seen as a lamination, $\mathcal{P}_n$ almost surely converges toward the Brownian triangulation $\mathcal{B}$, for the Hausdorff topology.

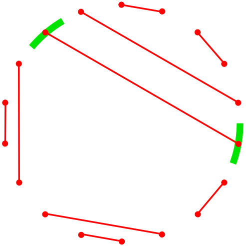
Inserting a chord in a noncrossing pair partition

Data: a noncrossing pair partition of size $2n$ and a $k \in \{0, 1, \ldots, 2n\}$





inserting a short chord



inserting a long chord

Growing algorithm

Algorithm

1. *Let $\tilde{\mathcal{P}}_1$ be the only pair partition of size 2.*
2. *Generate $\tilde{\mathcal{P}}_{n+1}$ from $\tilde{\mathcal{P}}_n$ as follows:*
    1. *choose an integer k uniformly at random in $\{0, 1, \ldots, 2n\}$;*
    2. *with probabilities $1/2$ - $1/2$, set $\tilde{\mathcal{P}}_{n+1}$ to be obtained from $\tilde{\mathcal{P}}_n$ by inserting at position k*
        - *either a short chord,*
        - *or a long chord.*

Growing algorithm

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1. *Let* $\tilde{\mathcal{P}}_1$ *be the only pair partition of size* 2.
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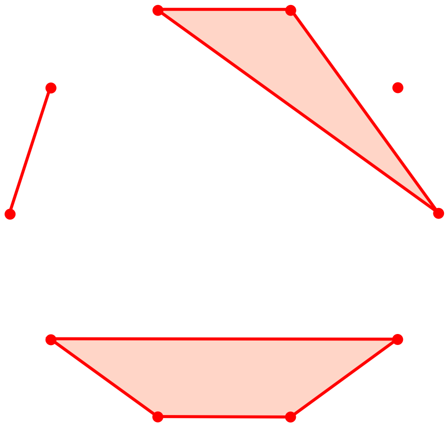
Encoding a noncrossing partition by a Dyck path



- Take a noncrossing partition of size n.

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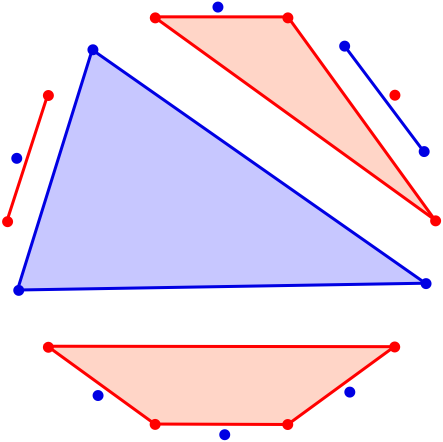

- Take a noncrossing partition of size n.
- Consider its Kreweras complement.

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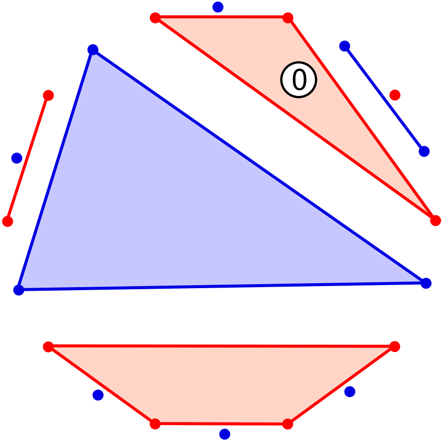

- Take a noncrossing partition of size n.
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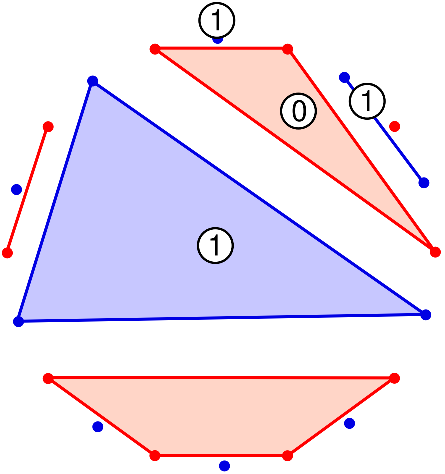
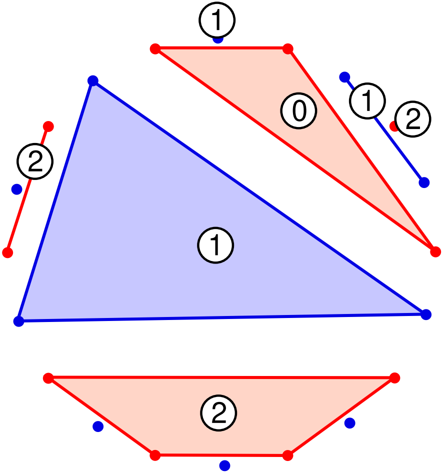
- Assign label 0 to the first block

Encoding a noncrossing partition by a Dyck path



- Take a noncrossing partition of size n.
- Consider its Kreweras complement.
- Assign label 0 to the first block
- Recursively assign label $\ell + 1$ to each not yet labeled neighbor of a block labeled ℓ.

Encoding a noncrossing partition by a Dyck path



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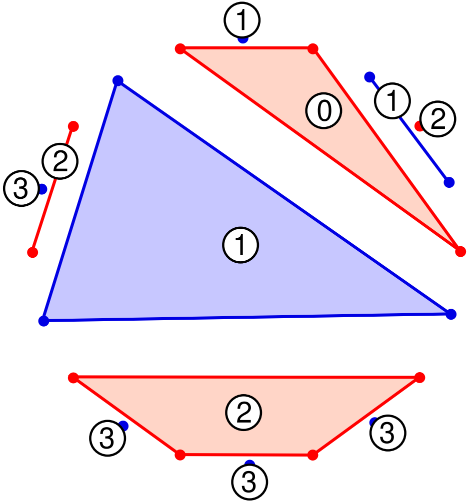

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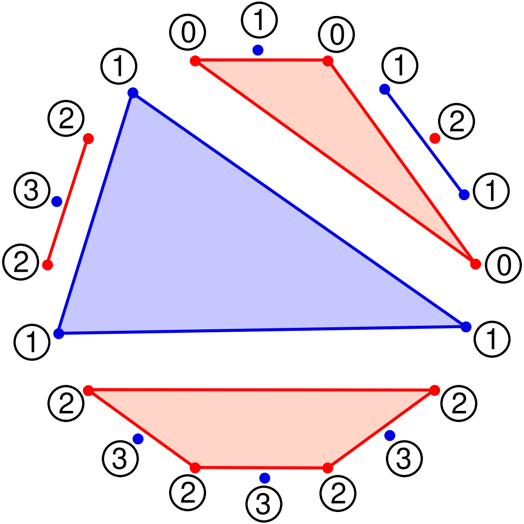

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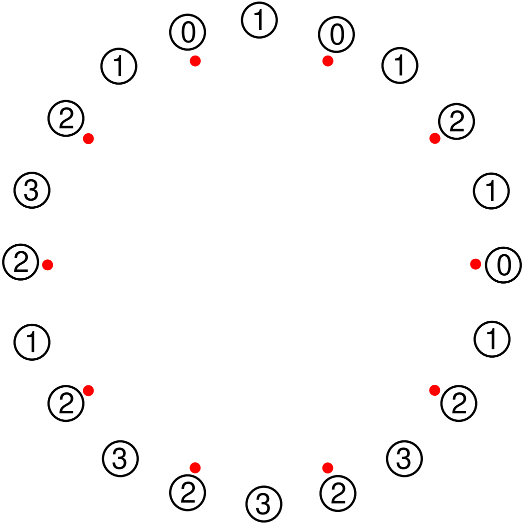
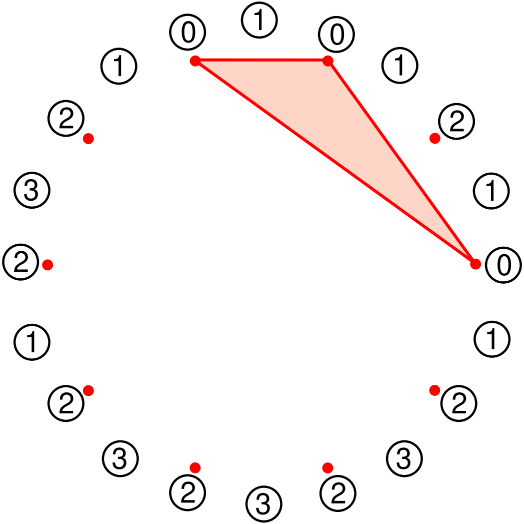
- Assign to each $2n$-th root of unity the label of its block.

Reverse mapping



- Take a $2n$-step Dyck path.

Reverse mapping



- Take a $2n$-step Dyck path.
- Put two n-th roots of unity in the same block when they share a label that is smaller than all the labels on an arc inbetween.

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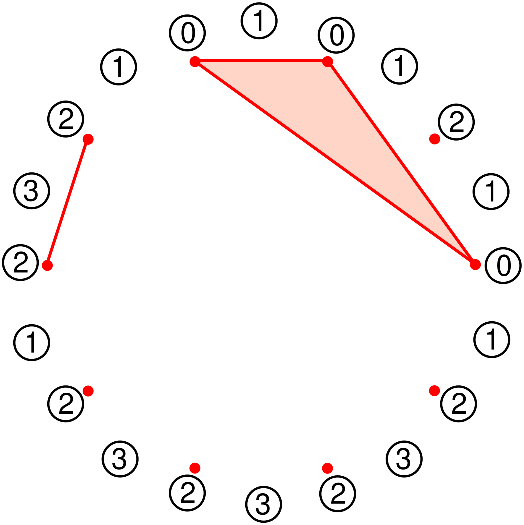

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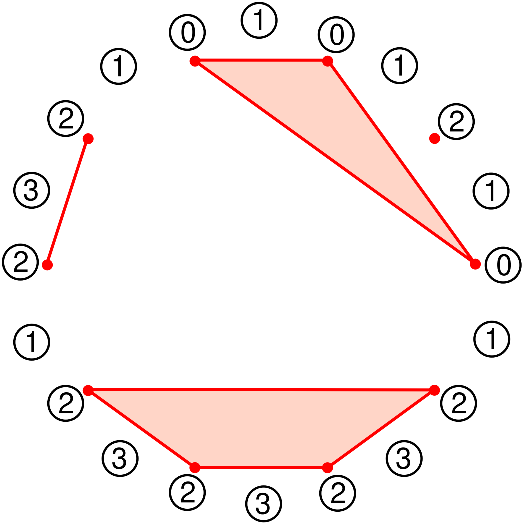

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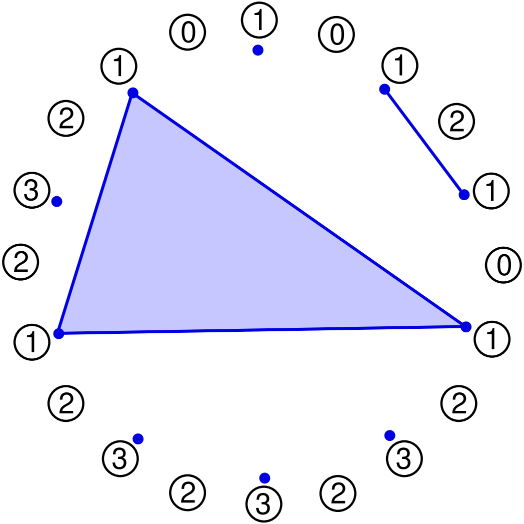

- Take a $2n$-step Dyck path.
- Put two n-th roots of unity in the same block when they share a label that is smaller than all the labels on an arc inbetween.
- The Kreweras complement is obtained by the same method with "odd" $2n$-th roots of unity.

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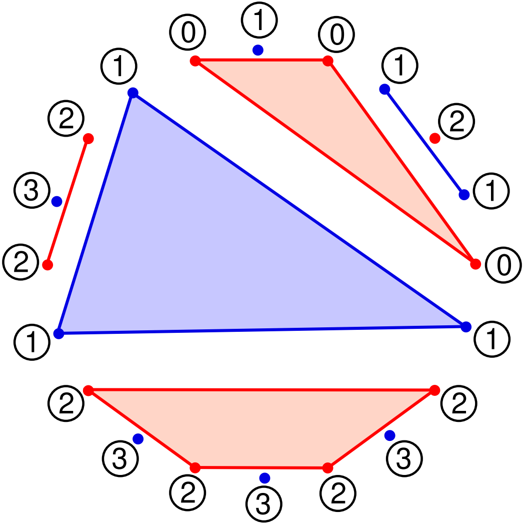

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Uniform noncrossing partitions

follows the general strategy of [Curien–Kortchemski '14]

- Let $\mathcal{P}_n$ be a uniform noncrossing partition of size n.
- Let $L_n : [0,1] \to \mathbb{R}_+$ be the encoding Dyck path ($L_n(k/2n)$: label of the k-th $2n$-th root of unity, and linear interpolation).
- By Kaigh's theorem (conditioned version of Donsker),

$$\left(\frac{L_n(s)}{\sqrt{2n}}\right)_{0\le s\le 1} \xrightarrow{(d)} (\mathbb{e}_s)_{0\le s\le 1} \cdot$$

- We apply Skorokhod's theorem and assume a.s. convergence.
- By compactness, $(\mathcal{P}_n)_n$ has accumulation points. Let $\mathcal{P}$ be one.
- We conclude by showing that $\mathcal{P} = \mathcal{B}$.

$\mathcal{P} = \mathcal{B}$

Reminder

$s \overset{\mathbb{e}}{\sim} t$ when $\mathbb{e}_s = \mathbb{e}_t = \min_{s\wedge t \le r \le s\vee t} \mathbb{e}_r$ $\qquad \mathcal{B} := \bigcup_{s \overset{\mathbb{e}}{\sim} t} \left[e^{2\mathrm{i}\pi s}, e^{2\mathrm{i}\pi t}\right]$

- As the local minimums of $\mathbb{e}$ on $(0,1)$ are distinct, if $s \overset{\mathbb{e}}{\sim} t$ with $s < t$, we can find even s_n, $t_n \in \{0, 2, 4, \dots, 2n\}$ such that $s_n < t_n$,

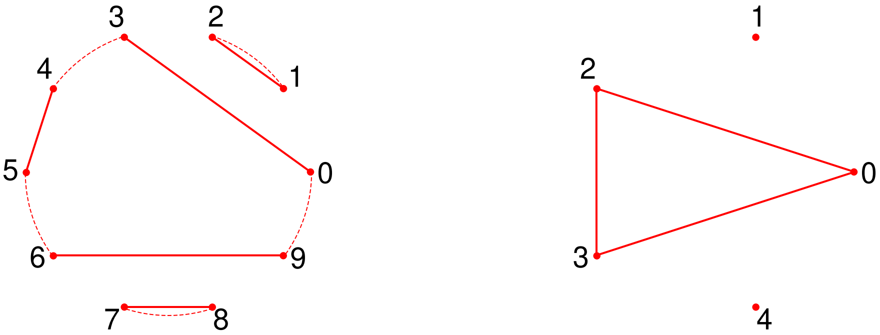
$$\frac{s_n}{2n} \to s\,, \quad \frac{t_n}{2n} \to t \quad \text{and} \quad L_n\Big(\frac{s_n}{2n}\Big) = L_n\Big(\frac{t_n}{2n}\Big) < \min_{\left[\frac{s_n+1}{2n},\frac{t_n-1}{2n}\right]} L_n\,.$$

- The chord $\left[\omega_{2n}^{s_n}, \omega_{2n}^{t_n}\right] \subseteq \mathcal{P}_n$, so that $[e^{2\mathrm{i}\pi s}, e^{2\mathrm{i}\pi t}] \subseteq \mathcal{P}$. Thus $\mathcal{B} \subseteq \mathcal{P}$.
- $\mathcal{B}$ is maximal for the inclusion relation, so that $\mathcal{B} = \mathcal{P}$.

Uniform noncrossing pair partitions: first approach

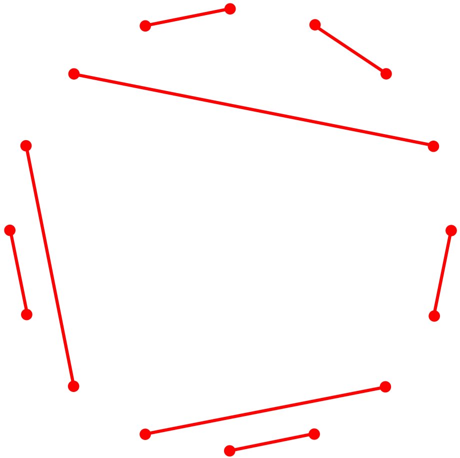
simple bijection

noncrossing pair partitions of size $2n \leftrightarrow$ noncrossing partitions of size n



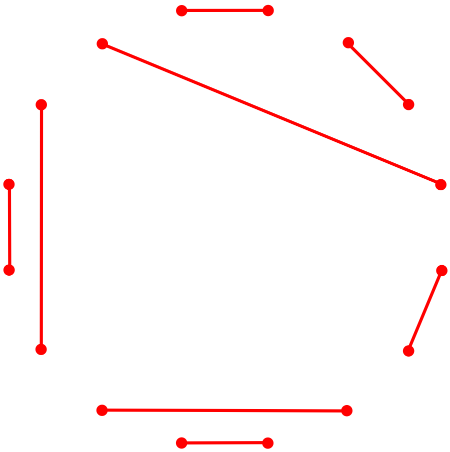
The corresponding laminations are at Hausdorff distance $\leq \pi/n$.

Uniform noncrossing pair partitions: direct proof



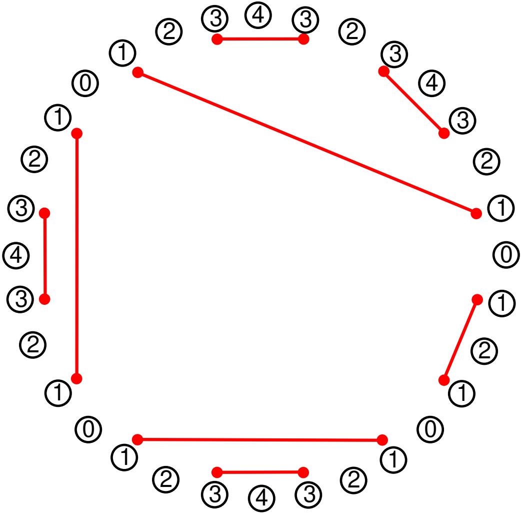
- Take a noncrossing partition of size $2n$.

Uniform noncrossing pair partitions: direct proof



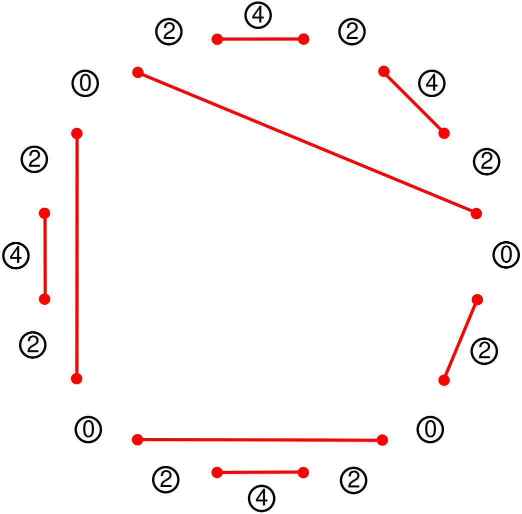
- Take a noncrossing partition of size $2n$.
- Rotate the picture by an angle of $-\pi/2n$.

Uniform noncrossing pair partitions: direct proof



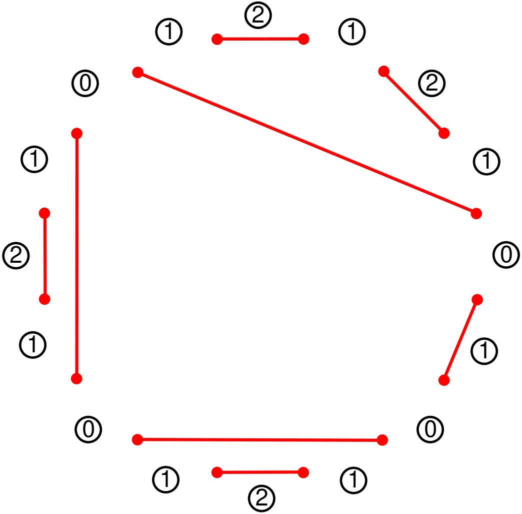
- Take a noncrossing partition of size $2n$.
- Rotate the picture by an angle of $-\pi/2n$.
- Encode the Kreweras complement by its Dyck path.

Uniform noncrossing pair partitions: direct proof



- Take a noncrossing partition of size $2n$.
- Rotate the picture by an angle of $-\pi/2n$.
- Encode the Kreweras complement by its Dyck path.
- The partition is a pair partition iff the Dyck path has no peaks at odd times; we may remove odd values.

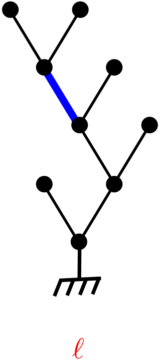
Uniform noncrossing pair partitions: direct proof



- Take a noncrossing partition of size $2n$.
- Rotate the picture by an angle of $-\pi/2n$.
- Encode the Kreweras complement by its Dyck path.
- The partition is a pair partition iff the Dyck path has no peaks at odd times; we may remove odd values.
- Divide by 2.

Rémy's bijection on binary trees

$$2 \quad (2n+1)\left|\{n\text{-node bin. trees}\}\right| = (n+2)\left|\{n+1\text{-node bin. trees}\}\right|$$



Rémy's bijection on binary trees

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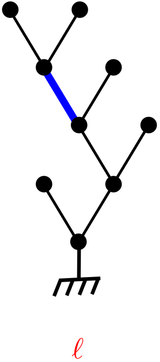

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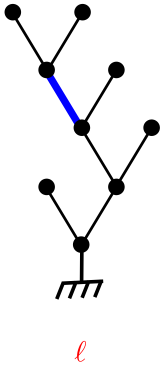
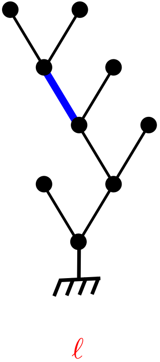
$$2\underbrace{(2n+1)}_{\text{edge}}\left|\{n\text{-node bin. trees}\}\right| = \underbrace{(n+2)}_{\text{leaf}}\left|\{n+1\text{-node bin. trees}\}\right|$$



Rémy's bijection on binary trees

$$\underbrace{2}_{\ell \text{ or } r}\underbrace{(2n+1)}_{\text{edge}}\big|\{n\text{-node bin. trees}\}\big| = \underbrace{(n+2)}_{\text{leaf}}\big|\{n+1\text{-node bin. trees}\}\big|$$



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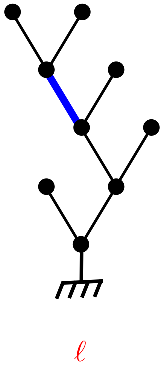
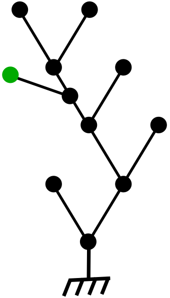


- Start from a binary tree with a marked edge and a mark ℓ (for *left*) or r (for *right*).

Rémy's bijection on binary trees

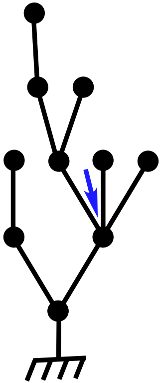
$$\underbrace{2}_{\color{red}{\ell \text{ or } r}}\,\underbrace{(2n+1)}_{\color{blue}{\text{edge}}}\left|\{n\text{-node bin. trees}\}\right| = \underbrace{(n+2)}_{\color{green}{\text{leaf}}}\left|\{n+1\text{-node bin. trees}\}\right|$$



- Start from a binary tree with a marked edge and a mark ℓ (for *left*) or r (for *right*).
- Add on the left or on the right of the edge a new edge and mark the leaf at its extremity.

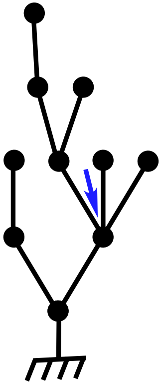
Rémy's bijection on plane trees

$$2\ (2n+1)\left|\{n\text{-edge trees}\}\right| = (n+2)\left|\{n+1\text{-edge trees}\}\right|$$



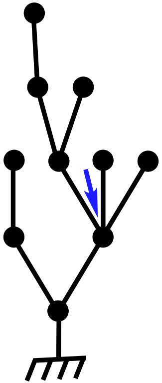
Rémy's bijection on plane trees

$$2\underbrace{(2n+1)}_{\text{corner}}\big|\{n\text{-edge trees}\}\big| = (n+2)\big|\{n+1\text{-edge trees}\}\big|$$



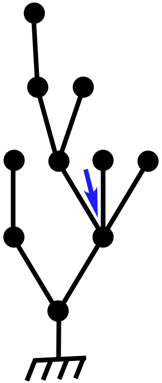
Rémy's bijection on plane trees

$$2\underbrace{(2n+1)}_{\text{corner}}\big|\{n\text{-edge trees}\}\big| = \underbrace{(n+2)}_{\text{vertex}}\big|\{n+1\text{-edge trees}\}\big|$$



Rémy's bijection on plane trees

$$\underbrace{2}_{a \text{ or } b}\ \underbrace{(2n+1)}_{\text{corner}}\Big|\{n\text{-edge trees}\}\Big| = \underbrace{(n+2)}_{\text{vertex}}\Big|\{n+1\text{-edge trees}\}\Big|$$



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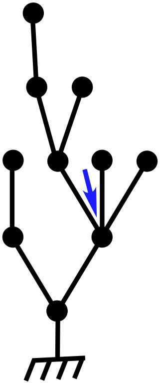

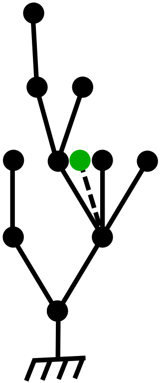

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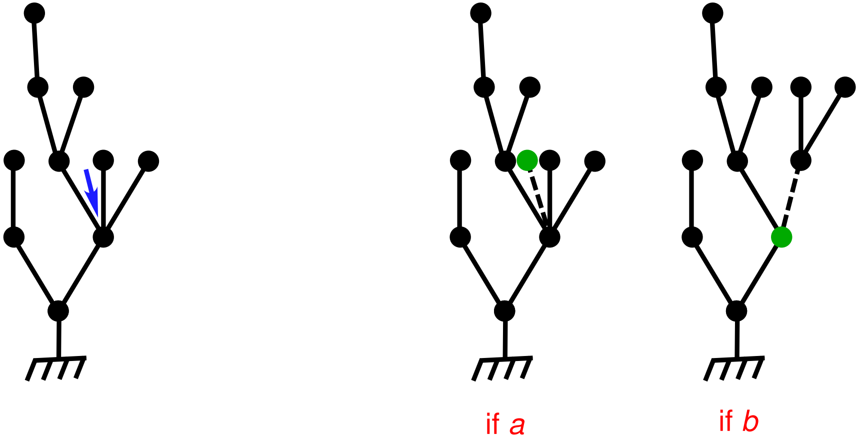
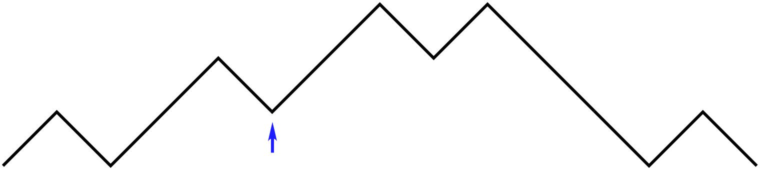


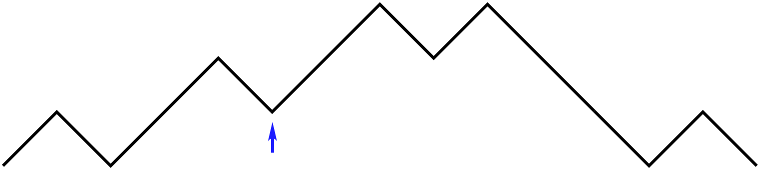
Rémy's bijection on Dyck paths

$$2 \quad (2n+1)\,\big|\{2n\text{-step paths}\}\big| = (n+2)\,\big|\{2n+2\text{-step paths}\}\big|$$



Rémy's bijection on Dyck paths

$$2\ \underbrace{(2n+1)}_{\text{time}}\left|\{2n\text{-step paths}\}\right| = (n+2)\left|\{2n+2\text{-step paths}\}\right|$$



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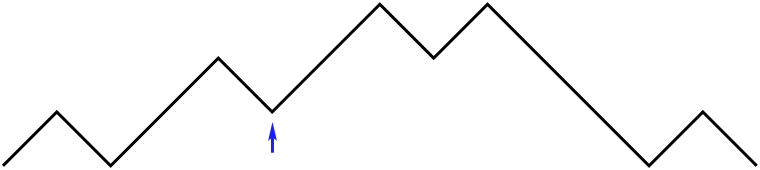
$$2\underbrace{(2n+1)}_{\text{time}}\big|\{2n\text{-step paths}\}\big| = \underbrace{(n+2)}_{\text{time}^\star}\big|\{2n+2\text{-step paths}\}\big|$$

time*: time s such that, for all $t > s$, $\inf_{s \le r \le t} D(r) < D(t)$



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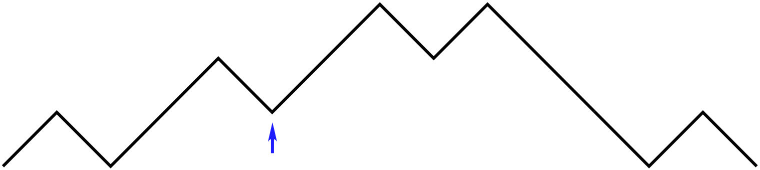

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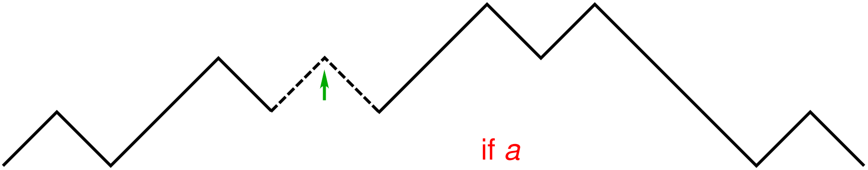

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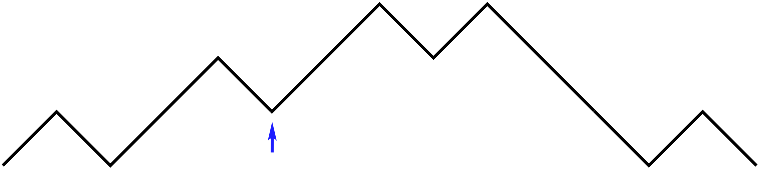

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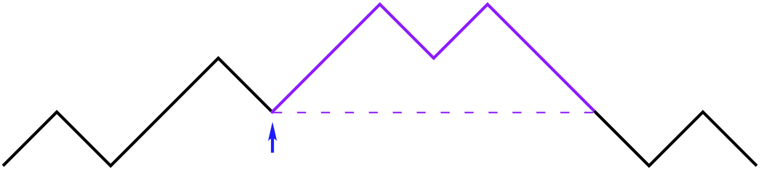

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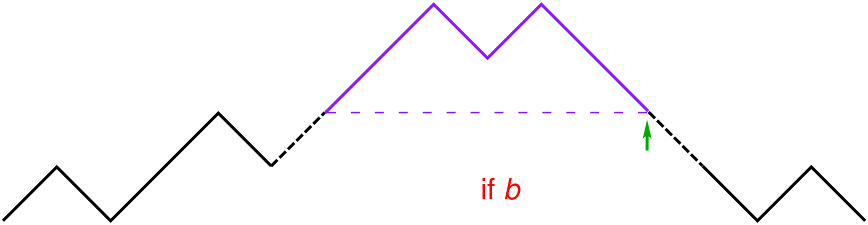

Rémy's algorithm on Dyck paths

Algorithm

1. *Let* D_1 *be the only* 2*-step Dyck path.*
2. *Generate* D_{n+1} *from* D_n *as follows:*
   1. *choose a time* k *uniformly at random in* $\{0, 1, \ldots, 2n\}$*;*
   2. *with probabilities* $1/2$ - $1/2$*, set the mark* m *to be* a *or* b*;*
   3. *set* D_{n+1} *as the image of* (D_n, k, m) *by Marchal's bijection.*

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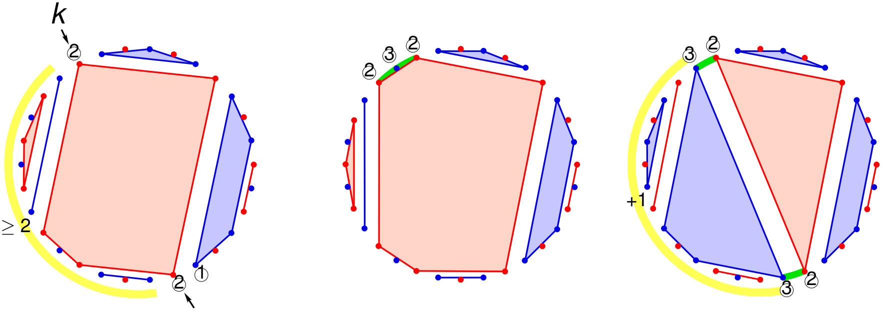
Theorem (Marchal '03)

The path D_n is a uniform $2n$-step Dyck path. Moreover, after linear interpolation,

$$\left(\frac{D_n(2n\,s)}{\sqrt{2n}}\right)_{0\le s\le 1} \to (\mathbb{e}_s)_{0\le s\le 1} \qquad a.s.$$

The algorithm on noncrossing partitions



inserting a vertex *slicing*

- Our algorithm on noncrossing partitions is the transcription of Marchal's algorithm, so that the convergence of the rescaled encoding Dyck path holds a.s. for this choice of sequence $(\mathcal{P}_n)_n$.
- Same thing for the algorithm on noncrossing pair partitions.