Random maps and Brownian surfaces

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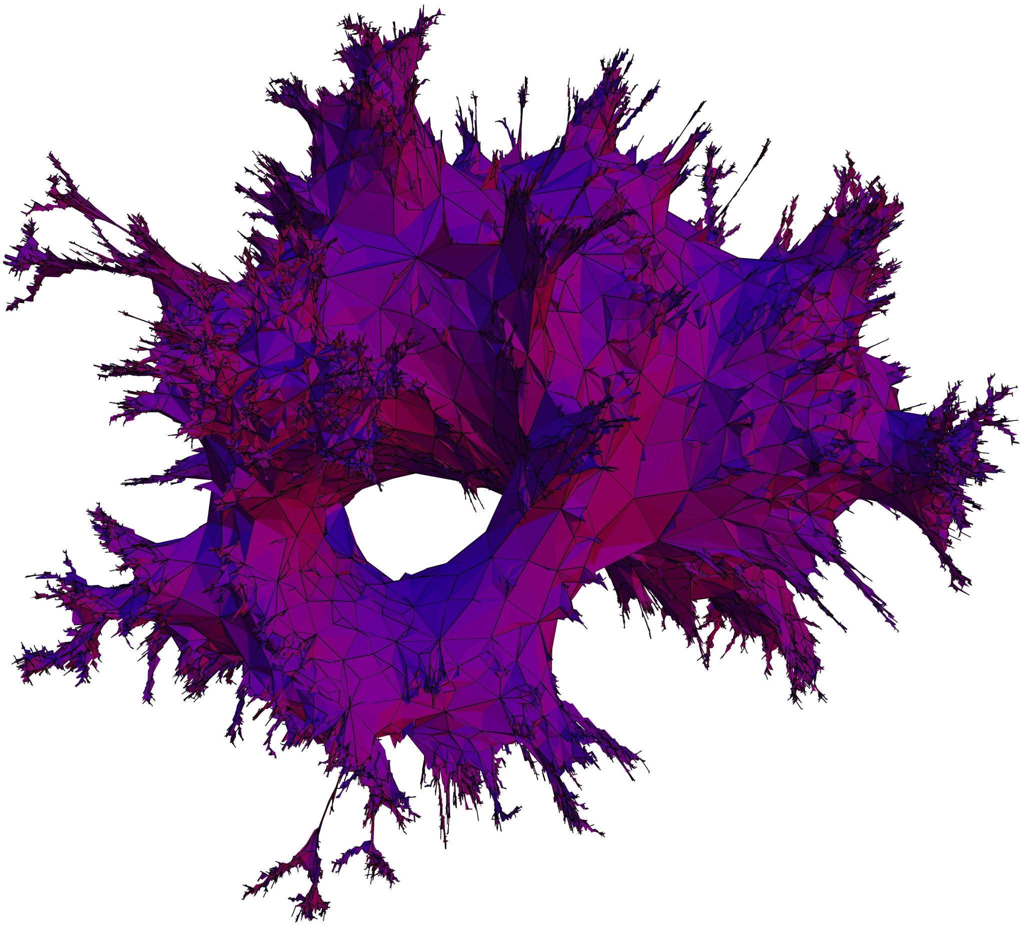
These lecture notes aim at covering the bases of the theory of scaling limits of random maps. For a more complete reference on the topic, the reader may consult [Mie14].



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1 Definition of maps

1.1 Maps

The precise definition of maps is a bit tedious but the objects are really rather intuitive. In order to easily grasp the concept, one might think of geographic maps, which give a slicing of some surface into geographical areas. For its part, the terminology rather comes from the world of polyhedra, with the notions of vertices, edges and faces. For the time being, we consider one of the following surfaces: the sphere, the torus, the double torus, the triple torus, etc. Recall that the *genus* of such a surface is its number of handles (see Figure 1).

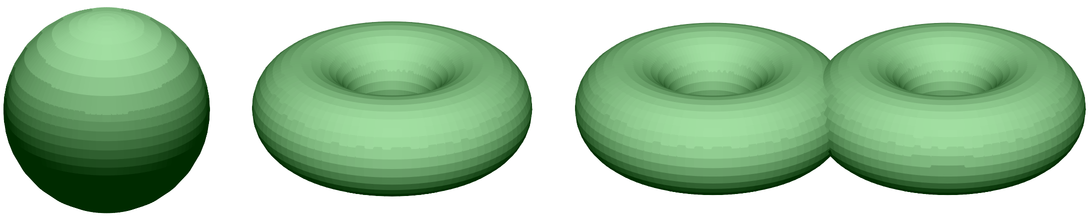

Figure 1: *The sphere (genre* 0*), the torus (genus* 1*) and the double torus (genus* 2*).*

We furthermore consider a finite graph[1], that is, a pair consisting in a finite set V of *vertices* and a finite multiset[2] $E \subseteq \{\{u, v\} : u, v \in V\}$ of *edges*. The *extremities* of the edge $\{u, v\}$ are the vertices u and v. Note that we do not exclude that several edges have the same extremities (we talk of *multiple edges*), neither that both extremities of some edges are equal (we say that such an

[1] As multiple edges and loops are allowed, the graphs we consider in these notes are sometimes refered to as *multiraph*, especially in the combonatorics community.

[2] A *multiset* is a set with repetitions allowed.

edge is a *loop*). An *embedding* of a graph (V, E) into a surface $\mathcal{S}$ is an injective function $f_V : V \to \mathcal{S}$ and a collection of continuous functions $f_e : [0,1] \to \mathcal{S}$, $e \in E$ such that

- ◇ for each edge $\{u, v\}$, $\{f_{\{u,v\}}(0), f_{\{u,v\}}(1)\} = \{f_V(u), f_V(v)\}$;
- ◇ for all $(s, e) \neq (s', e') \in (0,1) \times E$, we have $f_e(s) \neq f_{e'}(s')$.

In other words, it is an embedding of the vertices into the surface and the edges correspond to continuous curves linking the points of the surface that correpond to their extremities. Furthermore, the curves can only intersect at vertices. Next, we say that two embedings $(f_V, (f_e))$ and $(g_V, (g_e))$ of the same graph are *equivalent* if their exists an orientation-preserving homeomorphism φ of the underlying surface such that $g_V = \varphi \circ f_V$ and, for each $e \in E$, $g_e = \varphi \circ f_e$.

An embedding $(f_V, (f_e))$ of is called *cellular* if the connected components of $\mathcal{S} \setminus \bigcup_{e \in E} f_e([0,1])$ are open 2-cells, that is, homeomorphic to 2-dimensional open disks. We may finally give the proper definition of a map.

Definition 1. *A* map *is an equivalence class of cellular embeddings of a graph.*

The previous connected components are called the *faces* of the map. An edge given with an orientation is called a *half-edge*. We say that a face f is *incident* to a half-edge e (or that e is incident to f) if e is included in the boundary of f and is oriented in such a way that f lies to its left. The number of half-edges incident to a face is called its *degree*. The *genus* of a map is defined as the genus of the underlying surface. A map of genus 0 is called *plane*[3] or *spherical*; a map of genus 1 is called *toroidal*. Finally, all the maps we will consider will be *rooted*, that is, given with a distinguished half-edge, called the *root* of the map. See Figure 2 for an example.

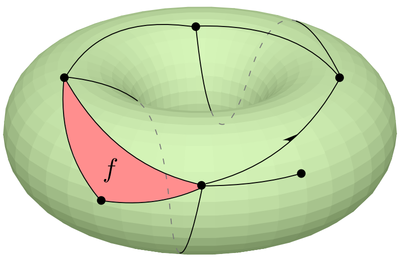

Figure 2: *Genus 1 map with 6 vertices, 9 edges and 3 faces. The highlighted face f has degree 3. The root is represented with a half arrowhead.*

We end this section with some basic notation, which will be used in these notes. For a map $\mathfrak{m}$, we let $V(\mathfrak{m})$ denote its set of vertices, $E(\mathfrak{m})$ its set of edges, $\vec{E}(\mathfrak{m})$ its set of half-edges, and $F(\mathfrak{m})$ its set of faces. For any half-edge e, we denote by $\bar{e}$ its reverse, as well as e^- and e^+ its origin and end. A *path* from $v \in V(\mathfrak{m})$ to $v' \in V(\mathfrak{m})$ is a finite sequence $(e_1, e_2, \ldots, e_k)$ of edges such that $e_i^+ = e_{i+1}^-$ for all $0 \leq i \leq k-1$. We denote by $d_\mathfrak{m}$ the *graph metric* on $\mathfrak{m}$ defined as follows: for any $v, v' \in V(\mathfrak{m})$, the distance $d_\mathfrak{m}(v, v')$ is the number of edges of any shortest path linking v to v'.

Finally, let us give the fundamental relation linking the number of vertices, edges and faces of a map.

Proposition 1 (Euler's characteristic formula)**.** *For any genus g-map $\mathfrak{m}$, one has the following identity:*

$$|V(\mathfrak{m})| - |E(\mathfrak{m})| + |F(\mathfrak{m})| = 2 - 2g\,.$$

[3]Technically, *planar* means "that **can be** embedded in the plane" whereas *plane* means "that **is** embedded in the plane." As a map is by definition embedded into a surface, it seems more logical to talk of *plane maps* rather than *planar maps*, although the latter term is widespread in the literature. Furthermore, we often consider a map embedded in the sphere or in the plane as being equivalent but, technically, this identification consists in distinguishing a face of the map as being the infinite part of the plane. As a result, in some references, plane maps are defined as maps with a distinguished face. In these notes, we only consider *rooted maps*, which come with a naturally distinguished face (the root face), so that the identification is licit.

1.2 Surfaces with a boundary

A *surface with a boundary* is a non empty Hausdorff topological space in which every point has an open neighborhood homeomorphic to some open subset of $\mathbb{R} \times \mathbb{R}_+$. Its *boundary* is the 1-dimensional manifold consisting of the points having a neighborhood homeomorphic to a neighborhood of $(0,0)$ in $\mathbb{R} \times \mathbb{R}_+$. Note that, in particular, a surface without boundary is a surface with a boundary whose boundary is empty. In these notes, we will only consider compact connected orientable surfaces with a boundary. By the classification theorem, they are characterized up to homeomorphisms by two nonnegative integers, the genus g and the number p of connected components of the boundary. We denote by $\mathbf{\Sigma}^{\partial}_{g,p}$ the unique (up to homeomorphism) compact orientable surface of genus g with p boundary components; it can be obtained from the compact orientable surface of genus g by removing p disjoint open disks whose boundaries are pairwise disjoint circles. See Figure 3.

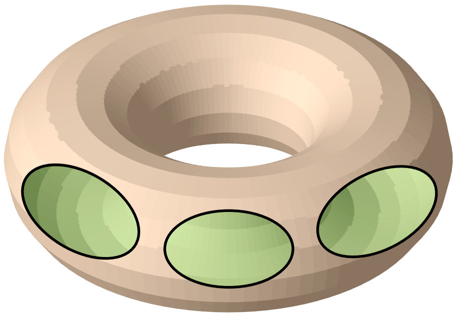

Figure 3: *The surface with a boundary* $\mathbf{\Sigma}^{\partial}_{1,3}$.

1.3 Quadrangulations with a boundary

For combinatorial reasons, we will restrict ourselves to bipartite maps: a map is called *bipartite* if its vertex set can be partitioned into two subsets such that no edge links two vertices of the same subset. For $p \geq 0$, a *quadrangulation with* p *boundary components* is a bipartite map having p distinguished faces $h_1, \ldots, h_p$ and whose other faces are all of degree 4. The distinguished faces will be called *external faces* or *holes*. The other faces will be called *internal faces*. For $n \in \mathbb{Z}_+$ and $\sigma = (\sigma^1, \ldots, \sigma^p) \in \mathbb{N}^p$ (with the convention that $\mathbb{N}^0 := \{\varnothing\}$), we define the set $\mathcal{Q}_{n,\sigma}$ of all genus g quadrangulations with p boundary components having n internal faces and such that h_i is of degree $2\sigma^i$, for $1 \leq i \leq p$. See Figure 4 for an example.

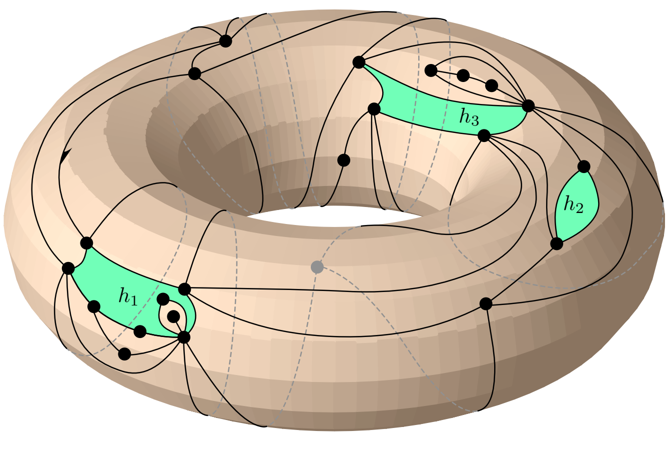

Figure 4: *A quadrangulation from* $\mathcal{Q}_{19,(4,1,2)}$ *in genus* 1*. The half arrowhead symbolizes the root.*

Beware that quadrangulations with a boundary are defined as embedded into a surface without boundary. Removing from the surface all the external faces of the map does not in general yield a surface with a boundary as the external faces may share vertices or edges or have a boundary that is not a simple curve (as h_1 on Figure 4 for instance). However, removing from every external face an open disk whose boundary is a circle yields a surface with a boundary.

2 Scaling limits of maps

2.1 The general concept of scaling limits

The notion of scaling limit is well known in probability theory. The general principle is as follows. We start with some class of combinatorial objects for which a notion of *volume* and a notion of *size* is given. As the volume tends to infinity, the idea is to renormalize the size in order to obtain an interesting object at the limit. More precisely, we pick an object "at random" among those of volume n in our combinatorial class. It may happen that, once the size properly renormalized, this random object converges in distribution toward a nontrivial limiting object as $n \to \infty$. A classical example is that of the standard random walk: we declare that the volume of a discrete walk is its number of steps and its size its final value. Then, this object has the Brownian motion as scaling limit: we pick uniformly at random an n-step walk with step-set $\{-1, +1\}$ and, after scaling the time by n and the space by $\sqrt{n}$, this random walk weakly converges toward a standard Brownian motion on $[0, 1]$, by Donsker's theorem.

The interest of this approach is twofold. On the one hand, the limiting object is a continuous object often interesting on its own, independently from the fact that it appears as the scaling limit of a discrete model. On the other hand, the study of the continuous model might reveal asymptotic properties of the discrete model, which can be hard to directly obtain. Moreover, the limiting object often possess a *universality* property: we obtain the same scaling limit for several different (but bearing some similarities) classes of combinatorial objects. This is for instance the case of Brownian motion, which appears as the scaling limit of any properly rescaled random walk, provided its step law is centered and of finite variance. We may also give the example of Aldous's Continuum Random Tree (CRT for short) [Ald91, Ald93], which is the scaling limit of many models of discrete random trees [DLG02].

2.2 A brief history

The natural problem of scaling limits of random maps has generated many studies in the last decade. The most natural setting is the following. We consider maps as metric spaces, endowed with their natural graph metric. We choose uniformly at random a map of "size" n in some class, rescale the metric by the proper factor, and look at the limit in the sense of the Gromov–Hausdorff topology (defined in Section 2.3). The size considered is often the number of faces of the map. From this point of view, the most studied class is the class of plane quadrangulations. The pioneering work of Chassaing and Schaeffer [CS04] revealed that the proper scaling factor in this case is $n^{-1/4}$. The problem was first addressed by Marckert and Mokkadem [MM06], who constructed a candidate limiting space called the *Brownian map*, and showed the convergence toward it in another sense. Le Gall [LG07] then showed the relative compactness of this sequence of metric spaces and that any of its accumulation points was almost surely of Hausdorff dimension 4. It is only recently that the solution of the problem was completed independently by Miermont [Mie13] and Le Gall [LG13], who showed that the scaling limit is indeed the Brownian map. This last step, however, is not mandatory in order to identify the topology of the limit: Le Gall and Paulin [LGP08], and later Miermont [Mie08], showed that any possible limit is homeomorphic to the 2-dimensional sphere.

This line of reasoning lead the way to several extensions. The first kind of extension is to consider other classes of plane maps. Actually, Le Gall already considered in [LG07] the classes of plane κ-angulations, for even $\kappa \geq 4$. In [LG13], he considered the classes of plane κ-angulations for $\kappa = 3$ and for even $\kappa \geq 4$ as well as the case of Boltzmann distributions on bipartite plane maps, conditioned on their number of vertices. Another extension is due to Addario-Berry and Albenque [ABA13], who consider simple plane triangulations and simple plane quadrangulations, that is, plane triangulations and quadrangulations without loops and multiple edges. Beltran et

Le Gall [BLG13] also studied plane quadrangulations without vertices of degree 1. Together with Jacob and Miermont [BJM14], we later added the case of plane maps conditioned on their number of edges, and Abraham [Abr16] considered the case of bipartite plane maps conditioned on their number of edges. More recently, Marzouk [Mar16] studied bipartite planar maps with a prescribed degree sequence. In all these cases, the limiting space is always the same Brownian map (up to a multiplicative constant): we say that the Brownian map is *universal* and we expect it to arise as the scaling limit of a lot more of natural classes of maps. A peculiar extension is due to Le Gall and Miermont [LGM11] who consider maps with large faces, forcing the limit to fall out of this universality class: they obtain so-called *stable maps*, which are related to stable processes.

Another kind of extension is to consider quadrangulations on a fixed surface that is no longer the sphere. The case of orientable surfaces of positive genus was the focus of [Bet10, Bet12] and the case of the disk was considered in [Bet15]. General quadrangulations with a boundary were studied in [Bet16]: this is the focus of these notes. The case of nonorientable quadrangulations without boundary was done in [CD17]. Finally, the more general case of nonorientable surfaces with a boundary is under study at the moment. In all the latter references, only convergence along subsequences is shown. In the case of the disk, the whole convergence is shown in [BM15] and, for general orientable quadrangulations with a boundary, it is the focus of [BM17].

The starting point of these studies is a powerful bijective encoding of the maps in the considered class by simpler objects. In the case of plane quadrangulations, the bijection in question is the so-called Cori–Vauquelin–Schaeffer bijection [CV81, Sch98, CS04] and the simpler objects are trees whose vertices carry integer labels satisfying some conditions. In the other cases, variants of this bijection are used [BDG04, CMS09, PS06, AB13] and the encoding objects usually have a more intricate combinatorial structure. Such an encoding will be presented in Section 3 for quadrangulations with a boundary.

2.3 The Gromov–Hausdorff topology

The Gromov–Hausdorff topology was introduced by Gromov [Gro99]. The idea is to compare metric spaces up to isometry. Roughly speaking, one wants to compare two metric spaces after isometrically embedding them as best as possible into a common metric space. First, the *Hausdorff distance* between two compact subsets A and B of a metric space $(\mathcal{X},\delta)$ is defined by

$$\delta_{\mathcal{H}}(A,B) \coloneqq \inf\left\{\varepsilon>0\,:\,A\subseteq B^{\varepsilon}\text{ et }B\subseteq A^{\varepsilon}\right\},$$

where, for any subset $X\subseteq\mathcal{X}$, we denote by $X^{\varepsilon}\coloneqq\{x\in\mathcal{X}\,:\,\delta(x,X)\le\varepsilon\}$ its ε-neighborhood. The Gromov–Hausdorff distance between two compact metric spaces $(\mathcal{X},\delta)$ and $(\mathcal{X}',\delta')$ is then defined by

$$d_{\mathrm{GH}}\big((\mathcal{X},\delta),(\mathcal{X}',\delta')\big)\coloneqq\inf\left\{\delta_{\mathcal{H}}\big(\varphi(\mathcal{X}),\varphi'(\mathcal{X}')\big)\right\},$$

where the infimum is taken over all isometric embeddings $\varphi:\mathcal{X}\to\mathcal{X}''$ and $\varphi':\mathcal{X}'\to\mathcal{X}''$ of $\mathcal{X}$ and $\mathcal{X}'$ into the same metric space $(\mathcal{X}'',\delta'')$.

Theorem 2 ([BBI01, Theorems 7.3.30 and 7.4.15])**.** *The Gromov–Hausdorff distance defines a metric on the set of isometry classes of compact metric spaces, making it a Polish space (separable and complete).*

The previous definition is not very easy to use in practice. It is often more convenient to use the following alternate definition. A *correspondence* between two metric spaces $(\mathcal{X},\delta)$ and $(\mathcal{X}',\delta')$ is a subset $\mathcal{R}\subseteq\mathcal{X}\times\mathcal{X}'$ whose projections are $\mathcal{X}$ and $\mathcal{X}'$. In other words, for all $x\in\mathcal{X}$, there is at least one $x'\in\mathcal{X}'$ for which $(x,x')\in\mathcal{R}$ and vice versa. The *distortion* of the correspondence $\mathcal{R}$ is defined by

$$\operatorname{dis}(\mathcal{R})\coloneqq\sup\left\{|\delta(x,y)-\delta'(x',y')|\,:\,(x,x'),(y,y')\in\mathcal{R}\right\}.$$

Proposition 3 ([BBI01, Theorem 7.3.25])**.** *The Gromov–Hausdorff distance between the two compact metric spaces $(\mathcal{X},\delta)$ and $(\mathcal{X}',\delta')$ is given by*

$$d_{\mathrm{GH}}(\mathcal{X},\mathcal{X}')=\frac{1}{2}\inf_{\mathcal{R}}\operatorname{dis}(\mathcal{R}),$$

where the infimum is taken over all correspondences between $\mathcal{X}$ and $\mathcal{X}'$.

2.4 Brownian surfaces

The aim of these notes is to obtain the following result.

Theorem 4 ([Bet16]). *Let us consider an integer $p \geq 0$, positive real numbers $\sigma_\infty^1, \dots, \sigma_\infty^p > 0$ and a sequence of p-uples $\sigma_n = (\sigma_n^1, \dots, \sigma_n^p) \in \mathbb{N}^p$ such that $\sigma_n^i/\sqrt{2n} \to \sigma_\infty^i$, for $1 \leq i \leq p$. Let $\mathfrak{q}_n$ be uniformly distributed over $\mathcal{Q}_{n,\sigma_n}$. Then, from any increasing sequence of integers, we may extract a subsequence $(n_k)_{k\geq 0}$ such that there exists a random metric space $(\mathfrak{q}_\infty^\sigma, d_\infty^\sigma)$ satisfying*

$$\left(V(\mathfrak{q}_{n_k}), \left(\frac{9}{8n_k}\right)^{1/4} d_{\mathfrak{q}_{n_k}}\right) \xrightarrow[k\to\infty]{(d)} (\mathfrak{q}_\infty^\sigma, d_\infty^\sigma)$$

in the sense of the Gromov–Hausdorff topology.

Remark. The constant $(8/9)^{1/4}$ is irrelevant in this statement. We chose to let it figure for the sake of consistency with other works and because of following definitions. This constant is inherent to the case of quadrangulations: we believe that the same statement should hold with the same limiting spaces for other classes of maps embedded in the same surface and satisfying mild conditions, up to modifying this constant.

This theorem generalizes the seminal result of Le Gall [LG07] in the spherical case ($(g,p) = (0,0)$) and our approach is widely inspired from his. As we will see later on, the spherical case is somehow degenerate in the sense that the encoding objects are slightly different. They are in fact easier to apprehend, but we will mostly leave this case out as it requires a special treatment.

Many questions arise at this point and some knowledge has been gathered on Brownian surfaces. We briefly discuss them here but these questions are all beyond the scope of these notes. The first result to have been obtained is the following:

Proposition 5 ([Bet16]). *Regardless of the choice of the sequence of integers, the previous limiting space $(\mathfrak{q}_\infty^\sigma, d_\infty^\sigma)$ is almost surely homeomorphic to $\mathbf{\Sigma}_{g,p}^\partial$, has Hausdorff dimension 4, and every of the p connected components of its boundary has Hausdorff dimension 2.*

This result generalizes the spherical case, which is due to Le Gall and Paulin [LGP08] and was shortly later reproved by Miermont [Mie08]. Notice that, although it seems reasonable that the limiting space will have genus at most g and at most p holes, it is not clear a priori that it will be homeomorphic to $\mathbf{\Sigma}_{g,p}^\partial$. We could imagine that some handles "disappear" or that some holes "merge" into a single hole. This does not happen; loosely speaking, this means that a uniform quadrangulation is sufficiently well spread over the surface, taking a macroscopic amount of space inbetween the holes and on every handle. Another noticeable fact is that the boundary of every hole is homeomorphic to a circle whereas, in the discrete picture, the holes do not in general have a simple curve as a boundary.

Clearly, the result of Theorem 4 is incomplete as one should wonder whether the extraction of subsequences is mandatory or not. The answer is no, but this requires a lot more work to obtain. Using two different approaches, Le Gall [LG13] and Miermont [Mie13] showed in major works that the extraction is not necessary in the spherical case. It is not necessary in the general case either; we showed this fact with Miermont in [BM15, BM17] thanks to an approach that roughly consists in cutting Brownian surfaces into elementary pieces of planar topology to which (a variant of) the result in the spherical case may be applied. This result validates the terminology of "Brownian surface." One might talk about *the Brownian torus* for instance.

See [Bet16] for a study of the geodesics to a given point. At the present time, there are no universality results except in the spherical case (as mentioned during Section 2.2) and in the case of the disk [BM15]. One might also wonder what happens if some σ_∞^i's are equal to 0. We believe that the convergence holds toward the Brownian surface that arises as the scaling limit of quadrangulations of the same surface without the corresponding holes. This was proved in [Bet15] in the case $g = 0$ and $p = 1$.

3 Encoding quadrangulations

Let us now present the bijection allowing to encode quadrangulations by simpler objects. Our description is a slight reformulation of the Bouttier–Di Francesco–Guitter bijection [BDG04]. We refer the reader to this reference for proofs (the proof is also outlined in Exercise 3). Throughout this section, n and p denote nonnegative integers and $\sigma = (\sigma^1, \dots, \sigma^p)$ a p-uple of positive integers.

3.1 From quadrangulations to labeled maps

Let $\mathfrak{q} \in \mathcal{Q}_{n,\sigma}$ be a quadrangulation and $v^\bullet \in V(\mathfrak{q})$ one of its vertices. We assign labels to the vertices of $\mathfrak{q}$ as follows: for every vertex $v \in V(\mathfrak{q})$, we set $\mathfrak{l}(v) := d_{\mathfrak{q}}(v^\bullet, v)$. Because $\mathfrak{q}$ is by definition bipartite, the labels of both ends of any edge differ by exactly 1. As a result, the internal faces can be of two types: the labels around the face are either d, $d+1$, $d+2$, $d+1$, or d, $d+1$, d, $d+1$ for some d. We add a new edge inside every internal face following the convention depicted on the left part of Figure 5.

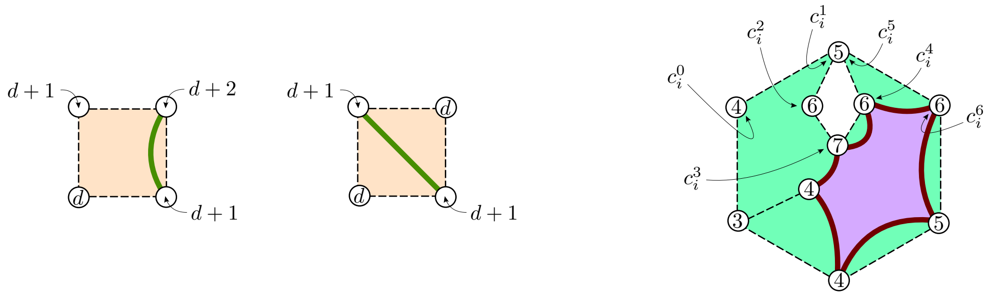

Figure 5: ***Left.*** *Adding a new edge inside an internal face.* ***Right.*** *Adding σ^i new edges inside the external face h_i. On this example, $\sigma^i = 6$.*

A *corner* is an angular sector delimited by two successive half-edges in the contour of a face. The vertex located at the end of the first half-edge is called the vertex *incident* to the corner. If c is a corner incident to a vertex v, we write $\mathfrak{l}(c) := \mathfrak{l}(v)$ with a slight abuse of notation. For each i, we let $c_i^0, c_i^1, \dots, c_i^{2\sigma^i-1}$ be the corners of h_i read in clockwise order, starting at an arbitrary corner (and we use the convention $c_i^{2\sigma^i} := c_i^0$). We link together in a cycle the corners c_i^k's such that $\mathfrak{l}(c_i^{k+1}) = \mathfrak{l}(c_i^k) - 1$, as shown on the right part of Figure 5. Note that, because $\mathfrak{l}(c_i^{k+1}) - \mathfrak{l}(c_i^k) \in \{-1, +1\}$, there are exactly σ^i such corners.

We then only keep the new edges we added and the vertices in $V(\mathfrak{q}) \backslash \{v^\bullet\}$. The object we obtain is a labeled map $\mathfrak{m}$ of genus g with $p+1$ faces. There is an obvious correspondence between the external faces of $\mathfrak{q}$ and p of the faces of $\mathfrak{m}$. Let $h_1, \dots, h_p$ also denote these faces in $\mathfrak{m}$. Note that, by construction, these faces all have a simple boundary and are of degrees $\sigma^1, \dots, \sigma^p$. Remark also that $v^\bullet$ lies within the remaining face of $\mathfrak{m}$, which we denote by $f^\bullet$.

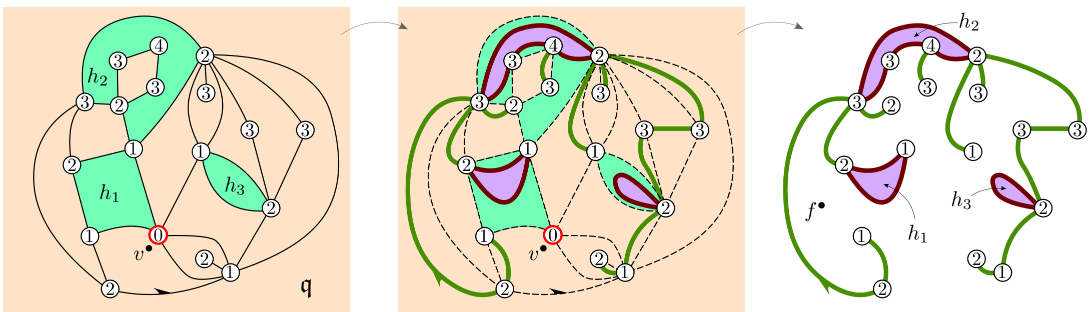

Figure 6: *The construction for a map in $\mathcal{Q}_{12,(2,4,1)}$ in the case $g = 0$.*

We root the map $\mathfrak{m}$ as follows. Let e be the only half-edge among the root of $\mathfrak{q}$ and its reverse such that $\mathfrak{l}(e^+) = \mathfrak{l}(e^-) + 1$, and let f be the face of $\mathfrak{q}$ that is incident to e. If f is an internal face, the root of $\mathfrak{m}$ is the half-edge corresponding to the edge we added in f, directed from e^+. If f is an external face, there are two new half-edges inside f starting from e^+; the root of $\mathfrak{m}$ is the one incident to $f^\bullet$. See Figure 7.

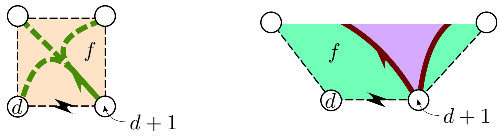

Figure 7: *Rooting the map $\mathfrak{m}$. On the picture, the two possible roots for $\mathfrak{q}$ yielding the same root for $\mathfrak{m}$ are shown.*

For each i, let $\vec{h}_i^1, \vec{h}_i^2, \ldots, \vec{h}_i^{\sigma^i}$ be the half-edges incident to h_i in $\mathfrak{m}$, read in counterclockwise order around it. The labels of $\mathfrak{m}$ satisfy the following:

- for $1 \le i \le p$ and $1 \le j \le \sigma^i$, we have $\mathfrak{l}(\vec{h}_i^{j-}) - \mathfrak{l}(\vec{h}_i^{j+}) \ge -1$;
- for any half-edge $e \in \vec{E}(\mathfrak{m})$ such that neither e nor its reverse $\bar{e}$ is incident to an h_i, we have $|\mathfrak{l}(e^+) - \mathfrak{l}(e^-)| \le 1$.

We will consider the labels of $\mathfrak{m}$ up to an additive constant: we write

$$[\mathfrak{l}] := \{v \in V(\mathfrak{m}) \mapsto \mathfrak{l}(v) + a : \ a \in \mathbb{Z}\}$$

the class of $\mathfrak{l}$ for this equivalence relation. We say that two faces are *adjacent* if there exist a half-edge incident to one and whose reverse is incident to the other. Let $\mathcal{M}_{n,\sigma}$ denote the set of genus g maps having $n + |\sigma|$ edges (where we write $|\sigma| := \sum_{i=1}^{p} \sigma^i$) and $p + 1$ faces denoted by $h_1, \ldots, h_p$, $f^\bullet$ such that, for all i, h_i has a simple boundary, is of degree σ^i and is not adjacent to any other h_j, and such that the root is not incident to any hole h_j. Note that any edge is forbidden to be incident to two different holes, but there may exist vertices that are incident to two or more holes. We also denote by $\mathfrak{M}_{n,\sigma}$ the set of pairs whose first coordinate lies in $\mathcal{M}_{n,\sigma}$ and whose second coordinate is an equivalence class of labeling functions on the vertices of the map satisfying the two previous itemized conditions.

Proposition 6. *The mapping $(\mathfrak{q}, v^\bullet) \mapsto (\mathfrak{m}, [\mathfrak{l}])$ is a two-to-one mapping from the set of quadrangulations in $\mathcal{Q}_{n,\sigma}$ carrying one distinguished vertex to the set $\mathfrak{M}_{n,\sigma}$.*

3.2 Reverse mapping

Let $(\mathfrak{m}, [\mathfrak{l}]) \in \mathfrak{M}_{n,\sigma}$. First, we add inside $f^\bullet$ a new vertex $v^\bullet$ with label

$$\mathfrak{l}(v^\bullet) := \min_{u \in V(\mathfrak{m})} \mathfrak{l}(u) - 1 \,.$$

Following the counterclockwise order around $f^\bullet$, we draw arcs linking every corner to the first subsequent corner that has a strictly smaller label. If no such corners exist, which means that the corner we are visiting has a minimal label, we draw the arc from the corner to the extra vertex $v^\bullet$. It is possible to draw these arcs in such a way that they do not cross each other or the edges of $\mathfrak{m}$.

When removing the edges of $\mathfrak{m}$, we are left with a quadrangulation $\mathfrak{q}$. Each hole h_i naturally corresponds to an external face of $\mathfrak{q}$, which we also denote by h_i. The root of $\mathfrak{q}$ is defined as the arc drawn from the corner preceding the root of $\mathfrak{m}$, oriented in one direction or the other. The two pre-images of $(\mathfrak{m}, [\mathfrak{l}])$ are the pairs $(\mathfrak{q}, v^\bullet)$, where $\mathfrak{q}$ is rooted in the two possible ways.

3.3 Further decomposition

A labeled map $(\mathfrak{m}, [\mathfrak{l}]) \in \mathfrak{M}_{n,\sigma}$ can be further decomposed into simpler objects: a *scheme*, which in some sense accounts for the homotopy type of $\mathfrak{m}$, and a collection of *forests* indexed by some half-edges of the scheme. The labeling function naturally gives rise to labels on the vertices of these forests as well as to bridges recording the labels on the cycles of the map $\mathfrak{m}$.

Remark 1. The case $(\mathfrak{m}, [\mathfrak{l}]) \in \mathfrak{M}_{n,\varnothing}$ in genus 0 is somehow degenerate. Indeed, in this case, $\mathfrak{m}$ is merely a plane tree and cannot be further decomposed in our sense. This is the original case treated in [LG07].

From now on, we exclude the case $(g, p) = (0, 0)$.

Remark 2. Up to a slight difference caused by the root, a scheme is sometimes called *kernel* in graph theory. We chose to stick with the terminology of scheme, which seems more common in the context of maps.

3.3.1 Decomposition of the map

Let us first focus on the map $\mathfrak{m}$ and keep the labels for later. We refer to Figure 10 for visual support. We iteratively remove from $\mathfrak{m}$ all its vertices of degree 1 that are not extremities of its root e_*. The set of vertices remaining at this point is called the *floor* of $\mathfrak{m}$. Among these vertices, some are called *nodes*: all vertices of degree 3 or more are nodes and,

- if e_*^- is of degree 1, then e_*^- is a node;
- if e_*^+ is of degree 1, then e_*^+ is a node;
- if neither e_*^- nor e_*^+ has degree 1, then e_*^- is a node.

On this map, the vertices that are not nodes are of degree 2 and are arranged into chains joining nodes. We define the map $\mathfrak{s}$ by replacing each of these chains by a single edge. The root of $\mathfrak{s}$ is defined as the edge replacing the chain that contains e_*, oriented in the same direction as e_*. The map $\mathfrak{s}$ is a scheme in the following sense.

Definition 2. *A genus g* scheme with p holes *is a genus g map with $p+1$ faces denoted by $h_1, \ldots, h_p$, $f^\bullet$, whose root is not incident to any h_j, and that satisfies the following conditions. For all i, h_i has a simple boundary and is not adjacent to any h_j. There may only be one vertex with degree 1 or 2: if it has degree 1, then it is an extremity of the root; if it has degree 2, then it is the origin of the root.*

Remark. A more conventional definition would be to forbid vertices of degree less than 2. Our choice of "keeping the root present" in the scheme is done for combinatorial reasons.

Let $\mathfrak{S}_p$ be the finite set of (genus g) schemes with p holes. For example, in genus $g = 0$, the set $\mathfrak{S}_1$ contains the three maps represented on Figure 8.

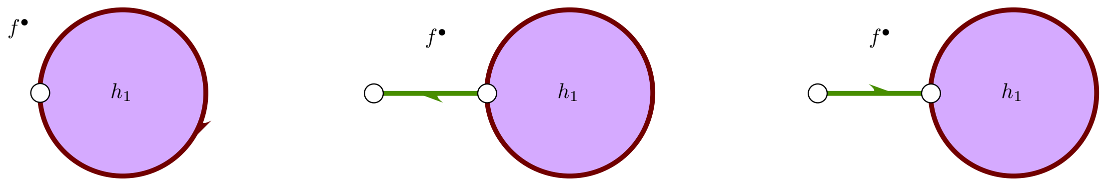

Figure 8: *The three elements of $\mathfrak{S}_1$ in genus $g = 0$.*

We will use the following formalism for forests.

Definition 3. *A* forest *of length $\xi \geq 1$ and mass $m \geq 0$ is an ordered family of ξ trees (which can be defined as plane one-faced maps) with total number of edges equal to m. Let $\mathcal{F}_\xi^m$ denote the set of these forests.*

It will be convenient to systematically add a $\xi+1$-th tree to a forest of $\mathcal{F}_\xi^m$ consisting of one single vertex. The *floor* of a forest is the set of the root vertices of its trees, with the extra vertex-tree included. In the drawings, we will also add extra edges linking the elements of the floor (as on Figure 10).

The half-edges of the scheme $\mathfrak{s}$ are of several different types. Let $\vec{H}_i(\mathfrak{s})$ be the set of half-edges incident to the hole h_i and let $\vec{F}(\mathfrak{s})$ be the set of half-edges that are not incident to any holes. We break the set $\vec{F}(\mathfrak{s})$ into two subsets: let $\vec{I}(\mathfrak{s}) := \{e \in \vec{F}(\mathfrak{s}) : \bar{e} \in \vec{F}(\mathfrak{s})\}$ and $\vec{B}(\mathfrak{s}) := \{e \in \vec{F}(\mathfrak{s}) : \bar{e} \notin \vec{F}(\mathfrak{s})\}$. See Figure 9. The letter I stands for "internal half-edges" and B stands for "boundary half-edges."

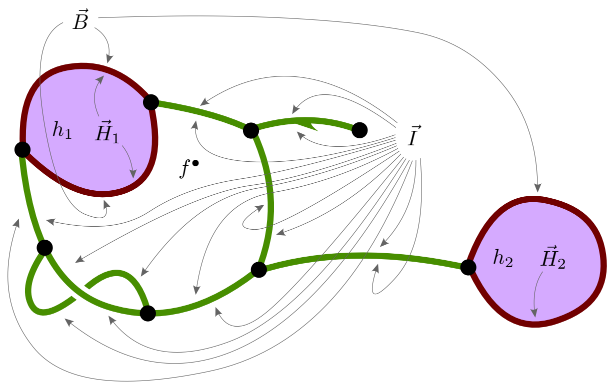

Figure 9: *Notation for the different types of half-edges of a genus 1 scheme with 2 holes. The set $\vec{F}(\mathfrak{s})$ is the disjoint union of $\vec{B}(\mathfrak{s})$ and $\vec{I}(\mathfrak{s})$.*

Every half-edge $e \in \vec{F}(\mathfrak{s})$ naturally corresponds to a forest $\mathfrak{f}^e$ defined as follows. Let $e' \in \vec{F}(\mathfrak{s})$ be the half-edge preceding e in the contour order of $f^\bullet$. By definition, the half-edges e and e' correspond to chains of half-edges in $\mathfrak{m}$: let $\hat{e}$ and $\hat{e}'$ be the last half-edges of these chains. The forest $\mathfrak{f}^e$ corresponds to the set of half-edges in $\mathfrak{m}$ visited between $\hat{e}'$ and $\hat{e}$ ($\hat{e}'$ excluded, $\hat{e}$ included) in the contour order of $f^\bullet$. See Figure 10.

For $e \in \vec{F}(\mathfrak{s})$, we denote by $\xi^e \geq 1$ and $m^e \geq 0$ the length and mass of $\mathfrak{f}^e$ (so that $\mathfrak{f}^e \in \mathcal{F}_{\xi^e}^{m^e}$).

Proposition 7. *The above decomposition provides a bijection between the set $\mathcal{M}_{n,\sigma}$ and the set of all pairs*

$$\left(\mathfrak{s}, (\mathfrak{f}^e)_{e\in\vec{F}(\mathfrak{s})}\right)$$

where $\mathfrak{s} \in \mathfrak{S}_p$ and such that there exist a collection of positive integers $(\xi^e)_{e\in\vec{E}(\mathfrak{s})}$ and a collection of nonnegative integers $(m^e)_{e\in\vec{F}(\mathfrak{s})}$ satisfying the following:

- *for all $e \in \vec{F}(\mathfrak{s})$, $\mathfrak{f}^e \in \mathcal{F}_{\xi^e}^{m^e}$;*
- *for all $e \in \vec{E}(\mathfrak{s})$, $\xi^{\bar{e}} = \xi^e$;*
- *for $0 \leq i \leq p$, $\sum_{e\in\vec{H}_i(\mathfrak{s})} \xi^e = \sigma^i$;*
- $\sum_{e\in\vec{F}(\mathfrak{s})} m^e + \frac{1}{2}\sum_{e\in\vec{E}(\mathfrak{s})} \xi^e = n + |\sigma|$.

3.3.2 Decomposition of the labeled map

Let us now take the labels into consideration. The terminology should become clear in a moment.

Definition 4. *We call* interior bridge *of length $\xi \geq 1$ from $a \in \mathbb{Z}$ to $b \in \mathbb{Z}$ a sequence of integers $(\mathfrak{b}(0), \ldots, \mathfrak{b}(\xi))$ such that $\mathfrak{b}(0) = a$, $\mathfrak{b}(\xi) = b$ and, for all $0 \leq i \leq \xi - 1$, we have $\mathfrak{b}(i+1) - \mathfrak{b}(i) \in \{-1, 0, 1\}$. We write $\mathfrak{I}_\xi(a, b)$ the set of these interior bridges.*

We call boundary bridge *of length $\xi \geq 1$ from $a \in \mathbb{Z}$ to $b \in \mathbb{Z}$ a sequence $(\mathfrak{b}(0), \ldots, \mathfrak{b}(\xi))$ of integers such that $\mathfrak{b}(0) = a$, $\mathfrak{b}(\xi) = b$ and, for all $0 \leq i \leq \xi - 1$, we have $\mathfrak{b}(i+1) - \mathfrak{b}(i) \geq -1$. We write $\mathfrak{B}_\xi(a, b)$ the set of these boundary bridges.*

Definition 5. *A* labeled forest *is a pair $(\mathfrak{f}, \ell)$ where $\mathfrak{f}$ is a forest and $\ell : V(\mathfrak{f}) \to \mathbb{Z}$ is a function satisfying the following:*

- ◇ *for all* u *lying in the floor of* $\mathfrak{f}$, $\ell(u) = 0$;
- ◇ *if* u *and* v *are linked by an edge, then* $|\ell(u) - \ell(v)| \leq 1$.

We denote by $\mathfrak{F}_\xi^m$ *the set of labeled forests of length* ξ *and mass* m.

There is a trivial one-to-one correspondence between the nodes of $\mathfrak{m}$ and the vertices of $\mathfrak{s}$ so that $\mathfrak{l}$ naturally gives a canonical labeling of the vertices of $\mathfrak{s}$ as follows. Let $v_* \in V(\mathfrak{s})$ be the origin of the root in $\mathfrak{s}$ and let $v \in V(\mathfrak{s})$. We denote by v'_* and $v' \in V(\mathfrak{m})$ the corresponding nodes in $\mathfrak{m}$ and we set $l^v := \mathfrak{l}(v') - \mathfrak{l}(v'_*)$. Let $e \in \vec{F}(\mathfrak{s})$. It naturally corresponds to a chain $e_1, e_2, \dots, e_{\xi^e}$ of half-edges in $\mathfrak{m}$. We define the bridge

$$\mathfrak{b}^e := \big(\mathfrak{l}(e_1^-) - \mathfrak{l}(v'_*), \mathfrak{l}(e_2^-) - \mathfrak{l}(v'_*), \dots, \mathfrak{l}(e_{\xi^e}^-) - \mathfrak{l}(v'_*), \mathfrak{l}(e_{\xi^e}^+) - \mathfrak{l}(v'_*)\big).$$

The constraints on $\mathfrak{l}$ show that, if $e \in \vec{I}(\mathfrak{s})$ then $\mathfrak{b}^e \in \mathfrak{I}_{\xi^e}(l^{e^-}, l^{e^+})$ and, if $e \in \vec{B}(\mathfrak{s})$ then $\mathfrak{b}^e \in \mathfrak{B}_{\xi^e}(l^{e^-}, l^{e^+})$. Moreover, the forest $\mathfrak{f}^e$ from last section naturally inherits from $\mathfrak{l}$ a labeling function $\mathfrak{l}^e$ defined as follows. Let $u \in V(\mathfrak{f}^e)$ and let $\rho \in V(\mathfrak{f}^e)$ be the root of the tree to which u belongs. These vertices correspond to two vertices u' and $\rho' \in V(\mathfrak{m})$. We set $\mathfrak{l}^e(u) := \mathfrak{l}(u') - \mathfrak{l}(\rho')$. See Figure 10.

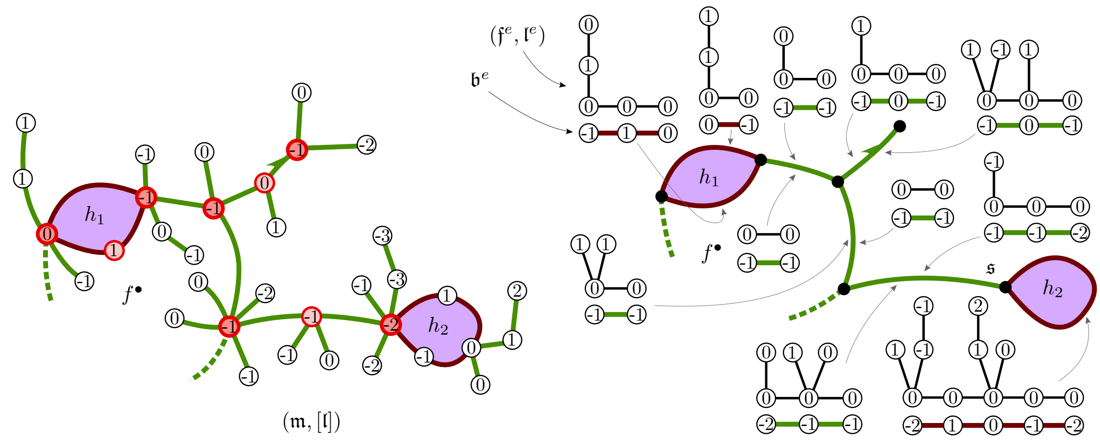

Figure 10: *Decomposition of a labeled map into a scheme* $\mathfrak{s}$, *a collection of labeled forests* $(\mathfrak{f}^e, \mathfrak{l}^e)_{e \in \vec{F}(\mathfrak{s})}$ *and a collection of bridges* $(\mathfrak{b}^e)_{e \in \vec{F}(\mathfrak{s})}$. *The bridges have different colors, depending on whether they are interior bridges or boundary bridges. On the left, the floor and 6 nodes of* $\mathfrak{m}$ *are represented in red and with thicker outlines.*

Proposition 8. *The above decomposition provides a bijection between the set* $\mathfrak{M}_{n,\sigma}$ *and the set of all triples*

$$\big(\mathfrak{s}, (\mathfrak{f}^e, \mathfrak{l}^e)_{e \in \vec{F}(\mathfrak{s})}, (\mathfrak{b}^e)_{e \in \vec{F}(\mathfrak{s})}\big)$$

where $\mathfrak{s} \in \mathfrak{S}_p$ *and such that there exist a collection of positive integers* $(\xi^e)_{e \in \vec{E}(\mathfrak{s})}$, *a collection of nonnegative integers* $(m^e)_{e \in \vec{F}(\mathfrak{s})}$ *and a collection of integers* $(l^v)_{v \in V(\mathfrak{s})}$ *satisfying the following:*

- ◇ *for all* $e \in \vec{F}(\mathfrak{s})$, $(\mathfrak{f}^e, \mathfrak{l}^e) \in \mathfrak{F}_{\xi^e}^{m^e}$;
- ◇ *for all* $e \in \vec{E}(\mathfrak{s})$, $\xi^{\bar{e}} = \xi^e$;
- ◇ $l^{v_*} = 0$, *where* v_* *is the origin of the root of* $\mathfrak{s}$;
- ◇ *for all* $e \in \vec{I}(\mathfrak{s})$, $\mathfrak{b}^e \in \mathfrak{I}_{\xi^e}(l^{e^-}, l^{e^+})$ *and* $\mathfrak{b}^{\bar{e}} = (\mathfrak{b}^e(\xi^e), \dots, \mathfrak{b}^e(0))$;
- ◇ *for all* $e \in \vec{B}(\mathfrak{s})$, $\mathfrak{b}^e \in \mathfrak{B}_{\xi^e}(l^{e^-}, l^{e^+})$;
- ◇ *for* $0 \leq i \leq p$, $\sum_{e \in \vec{H}_i(\mathfrak{s})} \xi^e = \sigma^i$;
- ◇ $\sum_{e \in \vec{F}(\mathfrak{s})} m^e + \frac{1}{2} \sum_{e \in \vec{E}(\mathfrak{s})} \xi^e = n + |\sigma|$.

Note that the three collections of integers are entirely determined by the triple of scheme, forests and bridges.

3.3.3 Encoding by real-valued functions

For $e \in \vec{F}(\mathfrak{s})$, we encode the labeled forest $(\mathfrak{f}^e, \mathfrak{l}^e)$ by its so-called *contour pair* (C^e, L^e) defined as follows. We see $\mathfrak{f}^e$ as a plane one-faced map with $2m^e + \xi^e$ edges (recall that we add a vertex-tree at the end and join by edges the elements of the floor). First, let $\mathfrak{f}^e(0), \mathfrak{f}^e(1), \dots, \mathfrak{f}^e(2m^e + \xi^e)$ be the vertices of $\mathfrak{f}^e$ read in counterclockwise order around the face, starting at the first corner of the first tree. The *contour function* $C^e : [0, 2m^e + \xi^e] \to \mathbb{R}_+$ and the *label function* $L^e : [0, 2m^e + \xi^e] \to \mathbb{R}$ are defined by

$$C^e(i) := d_{\mathfrak{f}^e}\big(\mathfrak{f}^e(i), \mathfrak{f}^e(2m^e + \xi^e)\big) \qquad \text{and} \qquad L^e(i) := \mathfrak{l}^e(\mathfrak{f}^e(i)), \qquad 0 \le i \le 2m^e + \xi^e,$$

and linearly interpolated between integer values (see Figure 11).

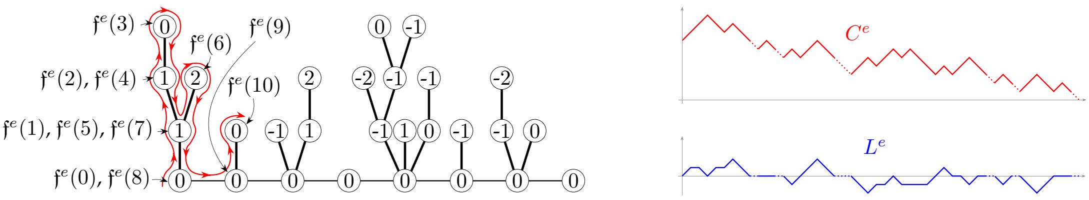

Figure 11: *The contour pair of a labeled forest from $\mathfrak{F}_7^{20}$. On the right, the paths are dashed on the intervals corresponding to edges linking elements of the floor.*

We also define the function $B^e : [0, \xi^e] \to \mathbb{R}$ by

$$B^e(i) := \mathfrak{b}^e(i), \qquad 0 \le i \le \xi^e,$$

and we linearly interpolate it between integer values. We will use the standard notation

$$\underline{X}(s) := \inf_{0 \le t \le s} X(t)$$

for the past infimum of any process X. Remark that the function

$$s \in [0, 2m^e + \xi^e] \mapsto L^e(s) + B^e\big(\xi^e - \underline{C}^e(s)\big) \tag{1}$$

records the labels up to an additive constant of the part of $\mathfrak{m}$ that corresponds to $\mathfrak{f}^e$. This will become useful in Section 4.3.

4 Scaling limit

From now on, we fix an integer $p \ge 0$ and a sequence $\sigma_n = (\sigma_n^1, \dots, \sigma_n^p) \in \mathbb{N}^p$ of p-uples such that

$$\sigma_{(n)}^i := \frac{\sigma_n^i}{\sqrt{2n}} \to \sigma_\infty^i \in (0, \infty), \qquad 1 \le i \le p.$$

We set $\sigma_\infty := (\sigma_\infty^1, \dots, \sigma_\infty^p)$ and $\sigma_{(n)} := \big(\sigma_{(n)}^1, \dots, \sigma_{(n)}^p\big)$ the rescaled version of σ_n. Recall also that the genus $g \ge 0$ is fixed and that $(g, p) \ne (0, 0)$.

Let $\mathfrak{q}_n$ be a random variable uniformly distributed over the set $\mathcal{Q}_{n,\sigma_n}$ and let $v_n^\bullet \in V(\mathfrak{q}_n)$ be one of its vertices chosen uniformly at random. Let $(\mathfrak{m}_n, [\mathfrak{l}_n]) \in \mathfrak{M}_{n,\sigma_n}$ be the image of $(\mathfrak{q}_n, v_n^\bullet)$ through the two-to-one mapping of Proposition 6 and let

$$\big(\mathfrak{s}_n, (\mathfrak{f}_n^e, \mathfrak{l}_n^e)_{e \in \vec{F}(\mathfrak{s}_n)}, (\mathfrak{b}_n^e)_{e \in \vec{F}(\mathfrak{s}_n)}\big)$$

be the decomposition of $(\mathfrak{m}_n, [\mathfrak{l}_n])$ appearing in Proposition 8. We let $(\xi_n^e)_{e \in \vec{E}(\mathfrak{s}_n)}$, $(m_n^e)_{e \in \vec{F}(\mathfrak{s}_n)}$ and $(l_n^v)_{v \in V(\mathfrak{s}_n)}$ be the three collections of integers from Proposition 8. For all e, we also denote by (C_n^e, L_n^e) the contour pair of $(\mathfrak{f}_n^e, \mathfrak{l}_n^e)$ as well as B_n^e the interpolation of $\mathfrak{b}_n^e$. Finally, we define the rescaled versions of these objects

$$m_{(n)}^e := \frac{2m_n^e + \xi_n^e}{2n}, \qquad \xi_{(n)}^e := \frac{\xi_n^e}{\sqrt{2n}}, \qquad l_{(n)}^v := \frac{l_n^v}{(8n/9)^{1/4}},$$

$$C^e_{(n)} := \left(\frac{C^e_n(2ns)}{\sqrt{2n}}\right)_{0\le s\le m^e_{(n)}}, \quad L^e_{(n)} := \left(\frac{L^e_n(2ns)}{(8n/9)^{1/4}}\right)_{0\le s\le m^e_{(n)}}, \quad B^e_{(n)} := \left(\frac{B^e_n(\sqrt{2n}\,s)}{(8n/9)^{1/4}}\right)_{0\le s\le \xi^e_{(n)}}.$$

The first goal of this section is to give the limit of the joint distribution of these processes. We first need to introduce the limiting processes.

4.1 Brownian bridges, first-passage Brownian bridges, and Brownian snake

We will work on the space $\mathcal{K} := \bigcup_{x\in\mathbb{R}_+} \mathcal{C}([0,x],\mathbb{R})$, endowed with the metric

$$d_{\mathcal{K}}(f,g) := |\zeta(f)-\zeta(g)| + \sup_{y\ge 0}\left|f\big(y\wedge\zeta(f)\big) - g\big(y\wedge\zeta(g)\big)\right|,$$

where $\zeta(f)$ denotes the only x such that $f\in\mathcal{C}([0,x],\mathbb{R})$.

We call *Brownian bridge* of length ξ from a to b a standard Brownian motion on $[0,\xi]$ started at a, conditioned on being at b at time ξ (see for example [Bil68, RY99, BCP03, Bet10]). We also call *first-passage Brownian bridge* of length m from a to $b<a$ a standard Brownian motion on $[0,m]$ started at a, and conditioned on hitting b for the first time at time m. We refer the reader to [Bet10] for a proper definition of this conditioning, as well as for some convergence results of the discrete analogs.

The so-called *Brownian snake's head* driven by a process $X\in\mathcal{C}([0,x],\mathbb{R})$ may be defined as the process $(X(s),Z(s))_{0\le s\le x}$, where, conditionally given X, the process Z is a centered Gaussian process with covariance function

$$\operatorname{cov}\big(Z(s),Z(s')\big) = \inf_{s\wedge s'\le t\le s\vee s'}\big(X(t)-\underline{X}(t)\big).$$

We refer to [LG99, DLG02, Bet10] for more details about this process.

4.2 Convergence of the encoding elements

In the limit, we will see that only the schemes that maximize the cardinal of $\vec{E}$ remain.

Definition 6. *A scheme is* dominant *if it has one vertex of degree exactly* 1 *and if all its other vertices are of degree exactly* 3*. Let $\mathfrak{S}^\star_p$ denote the set of (genus g) dominant schemes with p holes.*

For instance, the two right-most schemes of Figure 8 are the two elements of $\mathfrak{S}^\star_1$ in genus 0. Note that the degree 1-vertex in the previous definition is necessarily an extremity of the root and that the dominant schemes are the ones whose number of edges is maximal (see Exercise 2). Moreover, the condition on the degrees implies that every vertex of a dominant scheme is incident to at most one hole. In particular, the holes of a dominant scheme are well "separated" in the sense that their boundaries are pairwise disjoint simple loops, which are connected by some edges.

The compatibility condition on the previous collections of integers lead us to define, for a scheme $\mathfrak{s}$ with root e_*, the set $\mathcal{T}_{\mathfrak{s}}$ of triples

$$\Big((m^e)_{e\in\vec{F}(\mathfrak{s})}, (\xi^e)_{e\in\vec{E}(\mathfrak{s})}, (l^v)_{v\in V(\mathfrak{s})}\Big) \in \mathbb{R}_+^{\vec{F}(\mathfrak{s})}\times\mathbb{R}_+^{\vec{E}(\mathfrak{s})}\times\mathbb{R}^{V(\mathfrak{s})}$$

such that

- ⋄ $\sum_{e\in\vec{F}(\mathfrak{s})} m^e = 1$,
- ⋄ for all $e\in\vec{E}(\mathfrak{s})$, $\xi^{\bar{e}} = \xi^e$,
- ⋄ for $0\le i\le p$, $\sum_{e\in\vec{H}_i(\mathfrak{s})} \xi^e = \sigma^i_\infty$
- ⋄ $l^{e_*^-} = 0$.

We define the measure $\mathcal{L}_{\mathfrak{s}}$ on $\mathcal{T}_{\mathfrak{s}}$ as follows. For every $1 \leq i \leq p$, we distinguish a half-edge $\vec{h}_i \in \vec{H}_i(\mathfrak{s})$. We also consider an orientation $\check{I}(\mathfrak{s})$ of $\vec{I}(\mathfrak{s})$, that is, a subset of $\vec{I}(\mathfrak{s})$ containing exactly one half-edge among $\{e, \bar{e}\}$ for every $e \in \vec{I}(\mathfrak{s})$. The measure $\mathcal{L}_{\mathfrak{s}}$ is defined by

$$\mathcal{L}_{\mathfrak{s}}(\varphi) := \int \varphi\left((m^e)_{e\in\vec{F}(\mathfrak{s})}, (\xi^e)_{e\in\vec{E}(\mathfrak{s})}, (l^v)_{v\in V(\mathfrak{s})}\right) \prod_{e\in\vec{F}(\mathfrak{s})\setminus\{e_*\}} dm^e \prod_{e\in\Xi} d\xi^e \prod_{v\in V(\mathfrak{s})\setminus\{e_*^-\}} dl^v$$

for all measurable function φ, where $\Xi := \check{I}(\mathfrak{s}) \cup \bigcup_i \vec{H}_i(\mathfrak{s})\setminus\{\vec{h}_i\}$. Note that this definition does not depend on the choice of $\check{I}(\mathfrak{s})$.

We denote by p_a the density of a centered Gaussian variable with variance $a > 0$, as well as $-q_a$ its derivative:

$$p_a(x) := \frac{1}{\sqrt{2\pi\, a}} \exp\left(-\frac{x^2}{2a}\right) \qquad \text{and} \qquad q_a(x) = \frac{x}{a}\, p_a(x)\,, \quad x \in \mathbb{R}.$$

We let μ be the probability measure on $\bigcup_{\mathfrak{s}\in\mathfrak{S}^\star_{p,1}} \{\mathfrak{s}\} \times \mathcal{T}_{\mathfrak{s}}$ defined, for all measurable function ψ, by

$$\mu(\psi) := \frac{1}{\Upsilon} \sum_{\mathfrak{s}\in\mathfrak{S}^\star_p} \int_{\mathcal{T}_{\mathfrak{s}}} d\mathcal{L}_{\mathfrak{s}}\, \psi\left(\mathfrak{s}, \left((m^e), (\xi^e), (l^v)\right)\right) \prod_{e\in\vec{F}(\mathfrak{s})} q_{m^e}(\xi^e) \prod_{e\in\check{I}(\mathfrak{s})\cup\vec{B}(\mathfrak{s})} p_{(\kappa^e)^2\xi^e}\left(l^{e^+} - l^{e^-}\right),$$

where

$$\kappa^e := \begin{cases} 1 & \text{if } e \in \vec{I}(\mathfrak{s}_\infty) \\ \sqrt{3} & \text{if } e \in \vec{B}(\mathfrak{s}_\infty) \end{cases}$$

and

$$\Upsilon := \sum_{\mathfrak{s}\in\mathfrak{S}^\star_p} \int_{\mathcal{T}_{\mathfrak{s}}} d\mathcal{L}_{\mathfrak{s}} \prod_{e\in\vec{F}(\mathfrak{s})} q_{m^e}(\xi^e) \prod_{e\in\check{I}(\mathfrak{s})\cup\vec{B}(\mathfrak{s})} p_{(\kappa^e)^2\xi^e}\left(l^{e^+} - l^{e^-}\right)$$

is a normalization constant.

Proposition 9. *The random vector*

$$\left(\mathfrak{s}_n, \left(m^e_{(n)}\right)_{e\in\vec{F}(\mathfrak{s}_n)}, \left(\xi^e_{(n)}\right)_{e\in\vec{E}(\mathfrak{s}_n)}, \left(l^v_{(n)}\right)_{v\in V(\mathfrak{s}_n)}, \left(C^e_{(n)}, L^e_{(n)}\right)_{e\in\vec{F}(\mathfrak{s}_n)}, \left(B^e_{(n)}\right)_{e\in\vec{F}(\mathfrak{s}_n)}\right)$$

converges in distribution toward a random vector

$$\left(\mathfrak{s}_\infty, (m^e_\infty)_{e\in\vec{F}(\mathfrak{s}_\infty)}, (\xi^e_\infty)_{e\in\vec{E}(\mathfrak{s}_\infty)}, (l^v_\infty)_{v\in V(\mathfrak{s}_\infty)}, (C^e_\infty, L^e_\infty)_{e\in\vec{F}(\mathfrak{s}_\infty)}, (B^e_\infty)_{e\in\vec{F}(\mathfrak{s}_\infty)}\right)$$

whose law is defined as follows:

- ◇ *the vector* $\left(\mathfrak{s}_\infty, \left((m^e_\infty)_{e\in\vec{F}(\mathfrak{s}_\infty)}, (\xi^e_\infty)_{e\in\vec{E}(\mathfrak{s}_\infty)}, (l^v_\infty)_{v\in V(\mathfrak{s}_\infty)}\right)\right)$ *is distributed according to the probability measure* μ*;*
- ◇ *conditionally given this vector,*
  - *the processes* (C^e_∞, L^e_∞), $e \in \vec{F}(\mathfrak{s}_\infty)$ *and* B^e_∞, $e \in \check{I}(\mathfrak{s}_\infty) \cup \vec{B}(\mathfrak{s}_\infty)$ *are independent,*
  - *the process* (C^e_∞, L^e_∞) *has the law of a Brownian snake's head driven by a first-passage Brownian bridge of length* m^e_∞ *from* ξ^e_∞ *to* 0*,*
  - *the process* B^e_∞ *has the law of a Brownian bridge of length* ξ^e_∞ *from* $l^{e^-}_\infty$ *to* $l^{e^+}_\infty$*, multiplied by the factor* κ^e*,*
  - *the bridges are linked through the relation* $B^{\bar{e}}_\infty(s) = B^e_\infty(\xi^e_\infty - s)$, $0 \leq s \leq \xi^e_\infty$*, whenever* $e \in \vec{I}(\mathfrak{s}_\infty)$.

We will not prove this relatively intuitive proposition in these notes. It is easily obtained by the method provided in [Bet10] (see in particular Proposition 7 and Section 5, as well as [Bet15, Proposition 7]).

The factor κ^e accounts for the fact that the steps of boundary bridges have a larger variance than the steps of interior bridges (see Exercise 4). This seemingly harmless factor causes some difficulties for the technical estimates of [Bet15]. Note also that this proposition is the reason why the factor $(8/9)^{1/4}$ appears.

Let us emphasize that the limiting scheme $\mathfrak{s}_\infty$ is a.s. dominant and, as such, possesses the properties observed after Definition 6.

4.3 Proof of Theorem 4

The general strategy is borrowed from Le Gall's pioneering paper [LG07]. Recall that $\mathfrak{q}_n$ is uniformly distributed over $\mathcal{Q}_{n,\sigma_n}$, that $v_n^\bullet$ is uniformly distributed over $V(\mathfrak{q}_n)$ and that $(\mathfrak{m}_n,[\mathfrak{l}_n]) \in \mathfrak{M}_{n,\sigma_n}$ is the labeled map corresponding to $(\mathfrak{q}_n, v_n^\bullet)$.

Maps as quotients of integer intervals

We arrange the corners of the internal face of $\mathfrak{m}_n$ according to the contour order, starting from the first corner of the forest to which the root belongs[4]: this gives a natural (non injective) ordering of the vertices of $\mathfrak{m}_n$, which we write $\mathfrak{m}_n(0), \dots, \mathfrak{m}_n(2n+|\sigma_n|)$ with a slight abuse of notation. As the vertex set of $\mathfrak{m}_n$ corresponds to $V(\mathfrak{q}_n)\backslash\{v_n^\bullet\}$, this also provides an ordering $\mathfrak{q}_n(0), \dots, \mathfrak{q}_n(2n+|\sigma_n|)$ of the vertices of $\mathfrak{q}_n$. (The fact that $v_n^\bullet$ is left out will not be important when we take the scaling limit.)

For $i \le j$, we denote by $[\![i,j]\!] := [i,j] \cap \mathbb{Z} = \{i, i+1, \dots, j\}$. We endow $[\![0, 2n+|\sigma_n|]\!]$ with the pseudo-metric d_n defined by

$$d_n(i,j) := d_{\mathfrak{q}_n}\big(\mathfrak{q}_n(i), \mathfrak{q}_n(j)\big).$$

We define the equivalence relation $\sim_n$ on $[\![0, 2n+|\sigma_n|]\!]$ by declaring that $i \sim_n j$ if $\mathfrak{q}_n(i) = \mathfrak{q}_n(j)$, that is, if $d_n(i,j) = 0$. We let π_n be the canonical projection from $[\![0, 2n+|\sigma_n|]\!]$ to $[\![0, 2n+|\sigma_n|]\!]_{/\sim_n}$ and we slightly abuse notation by seeing d_n as a metric on the space $[\![0, 2n+|\sigma_n|]\!]_{/\sim_n}$, defined by $d_n\big(\pi_n(i), \pi_n(j)\big) := d_n(i,j)$. In what follows, we will always make the same abuse with every pseudo-metric. The metric space $\big([\![0, 2n+|\sigma_n|]\!]_{/\sim_n}, d_n\big)$ is then isometric to $\big(V(\mathfrak{q}_n)\backslash\{v_n^\bullet\}, d_{\mathfrak{q}_n}\big)$, which is at d_{GH}-distance at most 1 from the space $\big(V(\mathfrak{q}_n), d_{\mathfrak{q}_n}\big)$.

We extend d_n to non integer values by linear interpolation: for $s,t \in [0, 2n+|\sigma_n|]$,

$$d_n(s,t) := \underline{s}\,\underline{t}\, d_n(\lceil s\rceil, \lceil t\rceil) + \underline{s}\,\overline{t}\, d_n(\lceil s\rceil, \lfloor t\rfloor) + \overline{s}\,\underline{t}\, d_n(\lfloor s\rfloor, \lceil t\rceil) + \overline{s}\,\overline{t}\, d_n(\lfloor s\rfloor, \lfloor t\rfloor)\,, \tag{2}$$

where $\lfloor s\rfloor := \sup\{k \in \mathbb{Z},\, k \le s\}$, $\lceil s\rceil := \lfloor s\rfloor + 1$, $\underline{s} := s - \lfloor s\rfloor$ and $\overline{s} := \lceil s\rceil - s$. Beware that d_n is no longer a pseudo-metric on $[0, 2n+|\sigma_n|]$: indeed, $d_n(s,s) = 2\,\underline{s}\,\overline{s}\, d_n(\lceil s\rceil, \lfloor s\rfloor) > 0$ as soon as $s \notin \mathbb{Z}$. The triangle inequality, however, remains valid for all $s,t \in [0, 2n+|\sigma_n|]$.

We define the rescaled version: for $s,t \in [0,1]$, we let

$$d_{(n)}(s,t) := \left(\frac{9}{8n}\right)^{1/4} d_n\big((2n+|\sigma_n|)\,s, (2n+|\sigma_n|)\,t\big)\,, \tag{3}$$

so that

$$d_{\mathrm{GH}}\left(\left(\frac{1}{2n+|\sigma_n|}[\![0, 2n+|\sigma_n|]\!]_{/\sim_n}, d_{(n)}\right), \left(V(\mathfrak{q}_n), \left(\frac{9}{8n}\right)^{1/4} d_{\mathfrak{q}_n}\right)\right) \le \left(\frac{9}{8n}\right)^{1/4}. \tag{4}$$

Bound on the distances

The next step is to show the tightness of the processes $d_{(n)}$'s laws. For that matter, we use the following crucial bound:

$$d_n(i,j) \le d_n^\circ(i,j) := \mathfrak{l}_n\big(\mathfrak{m}_n(i)\big) + \mathfrak{l}_n\big(\mathfrak{m}_n(j)\big) - 2\max\left(\min_{k \in \overrightarrow{[\![i,j]\!]}} \mathfrak{l}_n\big(\mathfrak{m}_n(k)\big), \min_{k \in \overrightarrow{[\![j,i]\!]}} \mathfrak{l}_n\big(\mathfrak{m}_n(k)\big)\right) + 2$$

where we set

$$\overrightarrow{[\![i,j]\!]} := \begin{cases} [\![i,j]\!] & \text{if} \quad i \le j, \\ [\![i, 2n+|\sigma_n|]\!] \cup [\![0,j]\!] & \text{if} \quad j < i. \end{cases}$$

This bound is obtained by considering the path starting from the corner $\mathfrak{m}_n(i)$ and made of the successive arcs of the mapping of Section 3.2 until it reaches $v^\bullet$, as well as the similar path for $\mathfrak{m}_n(j)$. These paths are bound to meet at a vertex with label $l-1$, where

$$l := \min_{k \in \overrightarrow{[\![i,j]\!]}} \mathfrak{l}_n\big(\mathfrak{m}_n(k)\big).$$

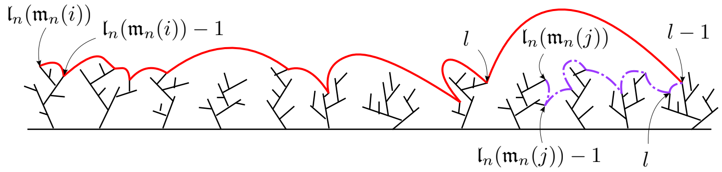

Figure 12: *The (red) plain path has length $\mathfrak{l}_n(\mathfrak{m}_n(i))-l+1$ and the (purple) dashed one has length $\mathfrak{l}_n(\mathfrak{m}_n(j))-l+1$. The concatenation of these paths gives a path of desired length.*

The claim easily follows; see Figure 12.

Similarly as above, we extend the definition of d_n° to $[0,2n+|\sigma_n|]$ by (2) and define its rescaled version $d_{(n)}^\circ$ by (3) (replacing each occurence of d_n with d_n°). In the end, we obtain that

$$d_{(n)}(s,t) \le d_{(n)}^\circ(s,t)\,, \qquad \text{for all } s,t\in[0,1]\,. \tag{5}$$

Expression of $d_{(n)}^\circ$ in terms of the encoding functions

We define the labeling function $\mathfrak{L}_n : [0,2n+|\sigma_n|] \to \mathbb{R}$ of the encoding map by,

$$\mathfrak{L}_n(i) \coloneqq \mathfrak{l}_n\big(\mathfrak{m}_n(i)\big) - \mathfrak{l}_n\big(\mathfrak{m}_n(0)\big), \qquad 0\le i\le 2n+|\sigma_n|,$$

and by linearly interpolating it between integer values. Its rescaled version is then defined by

$$\mathfrak{L}_{(n)} \coloneqq \left(\frac{\mathfrak{L}_n((2n+|\sigma_n|)\,s)}{(8n/9)^{1/4}}\right)_{0\le s\le 1},$$

so that

$$d_{(n)}^\circ(s,t) = \mathfrak{L}_{(n)}(s) + \mathfrak{L}_{(n)}(t) - 2\max\left(\min_{r\in\overrightarrow{[s,t]}} \mathfrak{L}_{(n)}(r),\ \min_{r\in\overrightarrow{[t,s]}} \mathfrak{L}_{(n)}(r)\right) + O\big(n^{\frac14}\big)\,,$$

where

$$\overrightarrow{[s,t]} \coloneqq \begin{cases} [s,t] & \text{if} \quad s\le t,\\ [s,1]\cup[0,t] & \text{if} \quad t<s. \end{cases}$$

Recall (1) expressing the labels of a forest in terms of the encoding functions. Writing $e_1,\dots,e_{\kappa_n}$ the half-edges of $\mathfrak{s}_n$ sorted according to its facial order, beginning with the root, we see that $\mathfrak{L}_{(n)}$ is the concatenation of the functions

$$\mathfrak{L}_{(n)}^e \coloneqq \Big(L_{(n)}^e(s) + B_{(n)}^e\big(\xi_{(n)}^e - \underline{C}_{(n)}^e(s)\big)\Big)_{0\le s\le m_{(n)}^e}, \qquad e\in\{e_1,\dots,e_{\kappa_n}\}\,.$$

Tightness of $d_{(n)}$

We apply Skorokhod's representation theorem and assume that the convergence of Proposition 9 holds almost surely. In particular, this entails that $\mathfrak{s}_n=\mathfrak{s}_\infty$ for n large enough. We only consider such n's from now on and denote by $e_1,\dots,e_\kappa$ the half-edges of $\mathfrak{s}_\infty$ sorted as above. Thanks to Proposition 9 and, as the concatenation is continuous from $(\mathcal{K},d_{\mathcal{K}})^2$ to $(\mathcal{K},d_{\mathcal{K}})$, we see that $\mathfrak{L}_{(n)}$ converges in $(\mathcal{K},d_{\mathcal{K}})$ toward a function $\mathfrak{L}_\infty$ defined as the concatenation of the functions

$$\mathfrak{L}_\infty^e \coloneqq \big(L_\infty^e(s) + B_\infty^e\big(\xi_\infty^e - \underline{C}_\infty^e(s)\big)\big)_{0\le s\le m_\infty^e}, \qquad e\in\{e_1,\dots,e_\kappa\}\,.$$

As a result, $d_{(n)}^\circ$ converges in $\big(\mathcal{C}([0,1]^2,\mathbb{R}),\|\cdot\|_\infty\big)$ toward d_∞° defined by

$$d_\infty^\circ(s,t) \coloneqq \mathfrak{L}_\infty(s) + \mathfrak{L}_\infty(t) - 2\max\left(\min_{r\in\overrightarrow{[s,t]}} \mathfrak{L}_\infty(r),\ \min_{r\in\overrightarrow{[t,s]}} \mathfrak{L}_\infty(r)\right), \qquad 0\le s,t\le 1\,.$$

[4] The choice of the starting point is arbitrary; starting at the first corner of a forest will simplify further expressions.

Lemma 10. *The sequence of the laws of the processes*

$$\left(d_{(n)}(s,t)\right)_{0\le s,t\le 1}$$

is tight in the space of probability measure on $\mathcal{C}([0,1]^2,\mathbb{R})$.

Proof. First observe that, for every s, s', t, $t'\in[0,1]$,

$$\left|d_{(n)}(s,t)-d_{(n)}(s',t')\right|\le d_{(n)}(s,s')+d_{(n)}(t,t')\le d^{\circ}_{(n)}(s,s')+d^{\circ}_{(n)}(t,t').$$

By Fatou's lemma, we have for every $k\in\mathbb{N}$ and $\delta>0$,

$$\limsup_{n\to\infty}\mathbb{P}\left(\sup_{|s-s'|\le\delta} d^{\circ}_{(n)}(s,s')\ge 2^{-k}\right)\le\mathbb{P}\left(\sup_{|s-s'|\le\delta} d^{\circ}_{\infty}(s,s')\ge 2^{-k}\right).$$

Since d°_{∞} is continuous and null on the diagonal, for $\varepsilon>0$, we may find $\delta_k>0$ such that, for n sufficiently large,

$$\mathbb{P}\left(\sup_{|s-s'|\le\delta_k} d^{\circ}_{(n)}(s,s')\ge 2^{-k}\right)\le 2^{-k}\varepsilon. \tag{6}$$

By taking δ_k even smaller if necessary, we may assume that the inequality (6) holds for all $n\ge 1$. Summing over $k\in\mathbb{N}$, we find that, for every $n\ge 1$,

$$\mathbb{P}\left(d_{(n)}\in\mathcal{K}_\varepsilon\right)\ge 1-\varepsilon,$$

where

$$\mathcal{K}_\varepsilon:=\left\{f\in\mathcal{C}([0,1]^2,\mathbb{R})\,:\,f(0,0)=0,\ \forall k\in\mathbb{N},\ \sup_{|s-s'|\wedge|t-t'|\le\delta_k}|f(s,t)-f(s',t')|\le 2^{1-k}\right\}$$

is a compact set. $\square$

Conclusion

Thanks to Lemma 10, there exist a subsequence $(n_k)_{k\ge0}$ and a function $d^{\sigma}_{\infty}\in\mathcal{C}([0,1]^2,\mathbb{R})$ such that

$$\left(d_{(n_k)}(s,t)\right)_{0\le s,t\le 1}\xrightarrow[k\to\infty]{(d)}\left(d^{\sigma}_{\infty}(s,t)\right)_{0\le s,t\le 1}. \tag{7}$$

As the functions $d_{(n)}$'s, the function d^{σ}_{∞} obeys the triangle inequality. And because $d^{\sigma}_{\infty}(s,s)\le d^{\circ}_{\infty}(s,s)=0$ for all $s\in[0,1]$, the function d^{σ}_{∞} is actually a pseudo-metric. We define the equivalence relation associated with it by saying that $s\sim_\infty t$ if $d^{\sigma}_{\infty}(s,t)=0$, and we set $\mathfrak{q}^{\sigma}_{\infty}:=[0,1]_{/\sim_\infty}$.

We claim that the convergence of Theorem 4 holds along the subsequence $(n_k)_{k\ge0}$. Using (4), we need to see that

$$d_{\mathrm{GH}}\left(\left(\frac{1}{2n+|\sigma_n|}[\![0,2n+|\sigma_n|]\!]_{/\sim_n},d_{(n)}\right),\left(\mathfrak{q}^{\sigma}_{\infty},d^{\sigma}_{\infty}\right)\right)\longrightarrow 0 \tag{8}$$

along the subsequence $(n_k)_{k\ge0}$. We will use Proposition 3 in order to show this fact. Recall that $\pi_n:[\![0,2n+|\sigma_n|]\!]\to[\![0,2n+|\sigma_n|]\!]_{/\sim_n}$ is the canonical projection. For $s\in[0,1]$, we let $\mathfrak{q}^{\sigma}_{\infty}(s)$ be the equivalence class of s in $\mathfrak{q}^{\sigma}_{\infty}$. The set

$$\mathcal{R}_n:=\left\{\left((2n+|\sigma_n|)^{-1}\,\pi_n\big(\lfloor(2n+|\sigma_n|)\,s\rfloor\big),\mathfrak{q}^{\sigma}_{\infty}(s)\right),\ s\in[0,1]\right\}$$

is clearly a correspondence between $\left((2n+|\sigma_n|)^{-1}[\![0,2n+|\sigma_n|]\!]_{/\sim_n},d_{(n)}\right)$ and $\left(\mathfrak{q}^{\sigma}_{\infty},d^{\sigma}_{\infty}\right)$. Its distortion is

$$\operatorname{dis}(\mathcal{R}_n)=\sup_{0\le s,t\le 1}\left|d_{(n)}\left(\frac{\lfloor(2n+|\sigma_n|)\,s\rfloor}{2n+|\sigma_n|},\frac{\lfloor(2n+|\sigma_n|)\,t\rfloor}{2n+|\sigma_n|}\right)-d^{\sigma}_{\infty}(s,t)\right|,$$

which, thanks to (7), tends to 0 along the subsequence $(n_k)_{k\ge0}$. By Proposition 3, we conclude that (8) holds.

4.4 Concluding remark: a bound on d_∞^σ

If we take the limit of the inequality (5) along the subsequence $(n_k)_{k\geq 0}$, we find $d_\infty^\sigma(s,t) \leq d_\infty^\circ(s,t)$. Because d_∞° does not satisfy the triangle inequality, we may improve this bound by considering the largest metric on $\mathfrak{q}_\infty^\sigma$ that is smaller than d_∞°: for all a and $b \in \mathfrak{q}_\infty^\sigma$, we have

$$d_\infty^\sigma(a,b) \leq d_\infty^*(a,b) := \inf \left\{ \sum_{i=0}^{k} d_\infty^\circ(s_i,t_i) \right\}$$

where the infimum is taken over all integer $k \geq 0$ and all sequences $s_0, t_0, s_1, t_1, \ldots, s_k, t_k$ satisfying $a = \mathfrak{q}_\infty^\sigma(s_0)$, for all $0 \leq i \leq k-1$, $t_i \sim_\infty s_{i+1}$, and $b = \mathfrak{q}_\infty^\sigma(t_k)$.

In fact, one can show that $d_\infty^\sigma = d_\infty^*$ and, as the definition of d_∞^* does not involve the extracted subsequence $(n_k)_{k\geq 0}$, one obtains the uniqueness of the limit, that is, the full convergence of Theorem 4. This is quite complicated and is the scope of [LG13, Mie13, BM15, BM17].

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Exercise sheet

1 Bipartite maps

1. Show that a map is bipartite if and only if every cycle made of edges in the map has an even number of edges.
2. Show that a plane map is bipartite if and only if all its faces have even degree. What about a positive genus map?

2 Schemes and dominant schemes

1. Using Euler's characteristic formula (Proposition 1), show that a genus g scheme with p holes has
   - $p+1$ faces;
   - at most $4g+2p$ vertices;
   - at most $6g+3p-1$ edges.
2. Let $\mathfrak{s} \in \mathfrak{S}_p$. Show that

$$\mathfrak{s} \in \mathfrak{S}_p^\star \qquad \Leftrightarrow \qquad \mathfrak{s} \text{ has } 4g+2p \text{ vertices} \qquad \Leftrightarrow \qquad \mathfrak{s} \text{ has } 6g+3p-1 \text{ edges} .$$

3 The encoding bijection

The aim of this exercise is to show that the mappings of Section 3 are as claimed. We denote by $\mathcal{Q}_{n,\sigma}^{\rightarrow\bullet}$ the set of pairs $(\mathfrak{q}, v^\bullet)$ where $\mathfrak{q} \in \mathcal{Q}_{n,\sigma}$, $v^\bullet \in V(\mathfrak{q})$ and the root of $\mathfrak{q}$ is such that its end is closer to $v^\bullet$ than its origin. We let $\Phi : \mathcal{Q}_{n,\sigma}^{\rightarrow\bullet} \to \mathfrak{M}_{n,\sigma}$ denote the restriction to $\mathcal{Q}_{n,\sigma}^{\rightarrow\bullet}$ of the mapping from Section 3.1 and by $\Psi : \mathfrak{M}_{n,\sigma} \to \mathcal{Q}_{n,\sigma}^{\rightarrow\bullet}$ the mapping from Section 3.2 that roots the resulting quadrangulation away from the origin of the initial root.

1. Let $(\mathfrak{m}, [\mathfrak{l}]) \in \mathfrak{M}_{n,\sigma}$ and $(\mathfrak{q}, v^\bullet) := \Psi(\mathfrak{m}, [\mathfrak{l}])$.
   (*a*) Show that $(\mathfrak{q}, v^\bullet) \in \mathcal{Q}_{n,\sigma}^{\rightarrow\bullet}$.
   (*b*) Show that the labels of $\mathfrak{m}$ are the distances to $v^\bullet$ in $\mathfrak{q}$ and conclude that $\Phi(\mathfrak{q}, v^\bullet) = (\mathfrak{m}, [\mathfrak{l}])$.
2. Let $(\mathfrak{q}, v^\bullet) \in \mathcal{Q}_{n,\sigma}^{\rightarrow\bullet}$ and $(\mathfrak{m}, [\mathfrak{l}]) = \Phi(\mathfrak{q}, v^\bullet)$.
   (*a*) We want to show that the embedded graph $\mathfrak{m}$ is a $p+1$-face map. To this end, we consider the map whose vertex set is $V(\mathfrak{q})$ and whose edges are the edges of $\mathfrak{q}$ together with the edges of $\mathfrak{m}$. We add inside each face of this map that is not a hole of $\mathfrak{m}$ a blue vertex and, for each edge of $\mathfrak{q}$, we add a blue dual edge linking the blue vertices of the two incident faces. We assign to each blue edge a label equal to the minimal label of the extremities of the edge it crosses.
      i. Consider a blue cycle and show that it necessarily circles around $v^\bullet$.
      ii. Deduce that $\mathfrak{m}$ is a map.
      iii. Using Euler's characteristic formula (Proposition 1), show that $(\mathfrak{m}, [\mathfrak{l}]) \in \mathfrak{M}_{n,\sigma}$.

(*b*) Show that the number of corners of the nonhole face of $\mathfrak{m}$ is equal to the number of edges of $\mathfrak{q}$ and finally that $\Psi(\mathfrak{m}, [\mathfrak{l}]) = (\mathfrak{q}, v^\bullet)$.

4 Interior bridges vs boundary bridges: what about this $\sqrt{3}$?

We will use in this exercise the following lemma:

Lemma 11 ([Bet10, Lemma 10])**.** *We consider a sequence $(X_k)_{k\geq 0}$ of i.i.d. centered integer-valued random variables with a moment of order q_0 for some $q_0 \geq 3$. We write $\eta^2 := \mathrm{Var}(X_1)$ its variance and h its maximal span*[5]*. We define $\Sigma_i := \sum_{k=0}^{i} X_k$ and still write Σ its linearly interpolated version. Let also $(r_n) \in \mathbb{Z}_+^{\mathbb{N}}$ and $(l_n) \in \mathbb{Z}^{\mathbb{N}}$ be two sequences of integers such that*

$$r_{(n)} := \frac{r_n}{n} \xrightarrow[n\to\infty]{} r \qquad \textit{and} \qquad l_{(n)} := \frac{l_n}{\eta\sqrt{n}} \xrightarrow[n\to\infty]{} l.$$

Let $(B_n(i))_{0\leq i\leq r_n}$ be the process whose law is the law of $(\Sigma_i)_{0\leq i\leq r_n}$ conditioned on the event $\{\Sigma_{r_n} = l_n\}$, which we suppose occurs with positive probability. Then, as n goes to infinity, the rescaled process

$$B_{(n)} := \left(\frac{B_n(ns)}{\eta\sqrt{n}}\right)_{0\leq s\leq r_{(n)}}$$

converges in law toward a Brownian bridge of length r from 0 to l, in the space $(\mathcal{K},\, d_{\mathcal{K}})$.

We consider two integer sequences (ξ_n) and (l_n) such that

$$\xi_{(n)} := \frac{\xi_n}{\sqrt{2n}} \xrightarrow[n\to\infty]{} \xi_\infty \qquad \text{and} \qquad l_{(n)} := \frac{l_n}{(8n/9)^{1/4}} \xrightarrow[n\to\infty]{} l_\infty\,.$$

1. Show that, if B_n is uniformly distributed over $\mathfrak{I}_{\xi_n}(0, l_n)$, then the process

$$B_{(n)} := \left(\frac{B_n(\sqrt{2n}\, s)}{(8n/9)^{1/4}}\right)_{0\leq s\leq \xi_{(n)}}$$

   converges in law toward a Brownian bridge of length ξ_∞ from 0 to l_∞.

2. Show that, if B_n is uniformly distributed over $\mathfrak{B}_{\xi_n}(0, l_n)$, then the process

$$B_{(n)} := \left(\frac{B_n(\sqrt{2n}\, s)}{(8n/9)^{1/4}}\right)_{0\leq s\leq \xi_{(n)}}$$

   converges in law toward a Brownian bridge of length ξ_∞ from 0 to l_∞, multiplied by the factor $\sqrt{3}$.

[5] The *maximal span* of an integer-valued random variable X is the greatest $h \in \mathbb{N}$ for which there exists an integer a such that a.s. $X \in a + h\mathbb{Z}$.