Geodesics in Brownian surfaces

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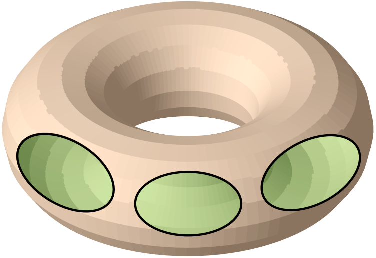
Surfaces with a boundary

Definition

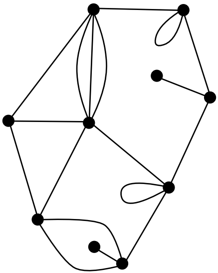
Let $\Sigma_{g,p}^{\partial}$ denote the surface with a boundary constructed by removing p open disks from the connected sum of g tori.

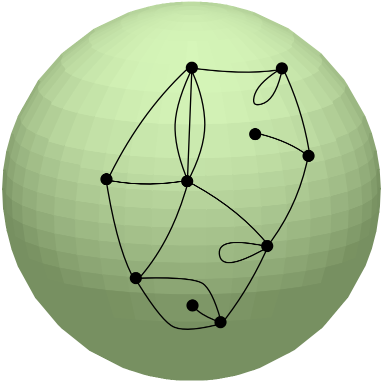


Classification theorem

Every compact, connected and orientable surface with a boundary is homeomorphic to a unique $\Sigma_{g,p}^{\partial}$.

Plane maps

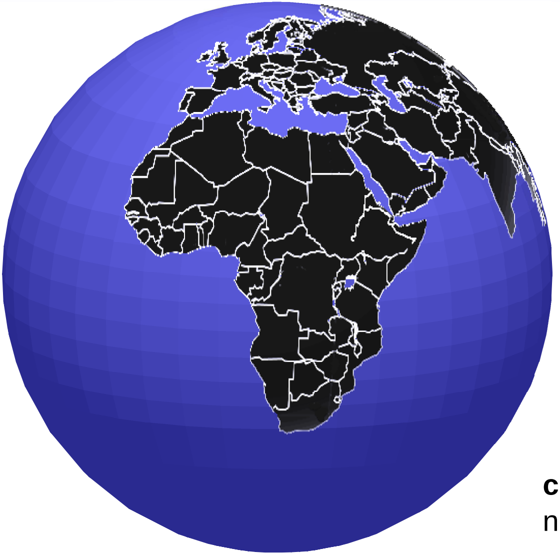




plane map: finite connected graph embedded in the sphere

faces: connected components of the complement

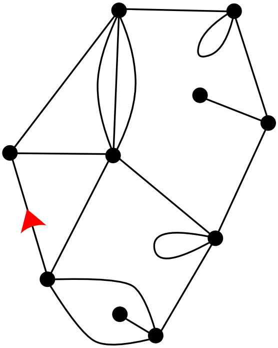
Example of plane map

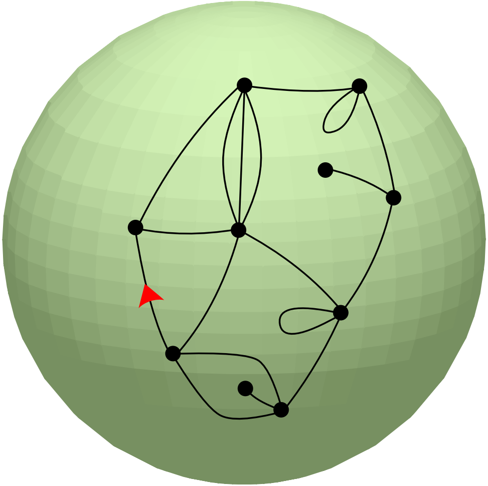


faces:
countries and
bodies of water

connected graph
no “enclaves”

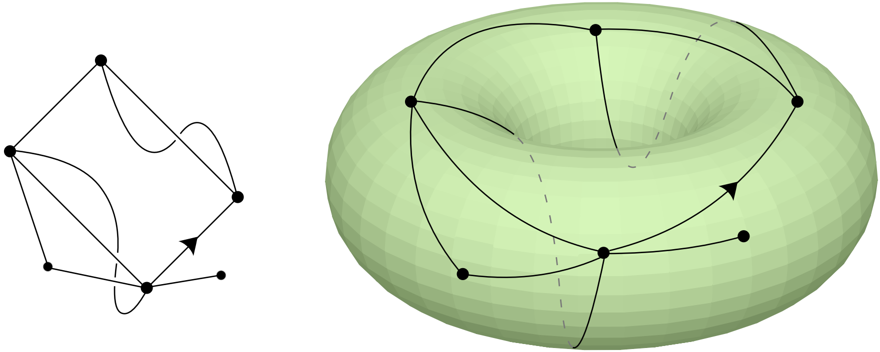
Rooted maps





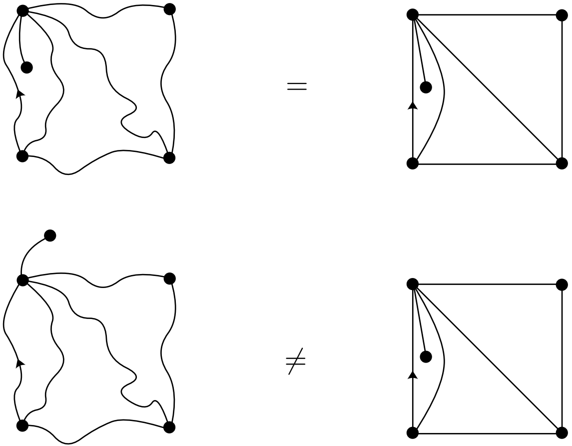
rooted map: map with one distinguished oriented edge

Genus g-maps

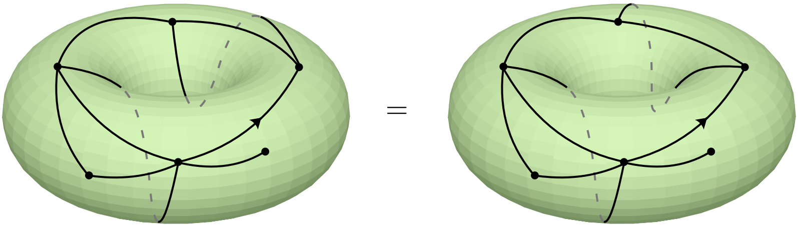


Genus g-map: graph embedded in the g-torus, in such a way that the faces are homeomorphic to disks

Edge deformation



More complicated deformation



maps are defined up to direct homeomorphism of the underlying surface

Quadrangulations with a boundary

Definition

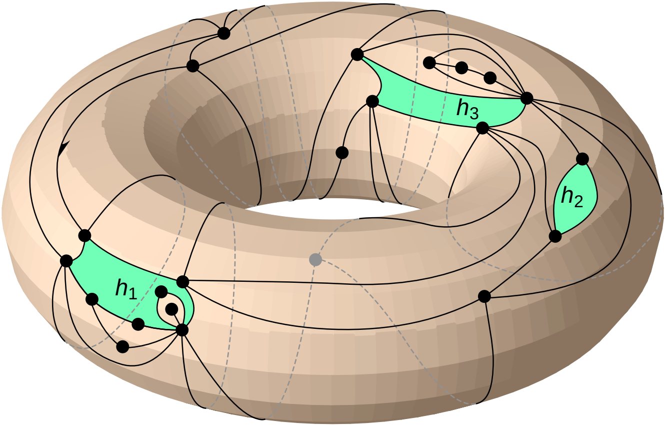
A quadrangulation with p holes is a rooted bipartite map with p distinguished faces $h_1, \ldots, h_p$ and whose other faces are of degree 4.

We fix the genus g and a p-uple $\sigma = (\sigma^1, \ldots, \sigma^p) \in \mathbb{N}^p$. We denote by $\mathcal{Q}_{n,\sigma}$ the set of genus g quadrangulations with p holes having n internal faces and such that h_i is of degree $2\sigma^i$, for $1 \le i \le p$.

Remark

A priori, a genus g quadrangulation with p holes is embedded in $\mathbf{\Sigma}^{\partial}_{g,0}$ and not in $\mathbf{\Sigma}^{\partial}_{g,p}$.

Example: element of $\mathcal{Q}_{19,(4,1,2)}$ in genus 1



Existence of scaling limits

- We denote by $V(\mathfrak{m})$ the vertex-set of $\mathfrak{m}$ and $d_{\mathfrak{m}}$ the graph metric.
- We work in fixed genus g.

Theorem (Le Gall '07 $[(g,p)=(0,0)]$, B. '14 $[(g,p)\neq(0,0)]$)

Let $p \in \mathbb{N}$, $\sigma_\infty^1, \ldots, \sigma_\infty^p > 0$ and $\sigma_n = (\sigma_n^1, \ldots, \sigma_n^p) \in \mathbb{N}^p$ be a sequence such that $\sigma_n^i/\sqrt{2n} \to \sigma_\infty^i$ for $1 \le i \le p$. Let also $\mathfrak{q}_n$ be uniformly distributed over $\mathcal{Q}_{n,\sigma_n}$. Then, from any increasing sequence of integers, we may extract a subsequence $(n_k)_{k\ge0}$ such that there exists a compact random metric space $(\mathfrak{q}_\infty^\sigma, d_\infty^\sigma)$ satisfying

$$\left(V(\mathfrak{q}_{n_k}), \left(\frac{9}{8n_k}\right)^{1/4} d_{\mathfrak{q}_{n_k}}\right) \xrightarrow[k\to\infty]{(d)} (\mathfrak{q}_\infty^\sigma, d_\infty^\sigma)$$

in the sense of the Gromov–Hausdorff topology.

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in the sense of the Gromov–Hausdorff topology.

Topology and Hausdorff dimension

Definition

We call Brownian surface the metric space $M_\sigma^{(g)}$ from the previous theorem.

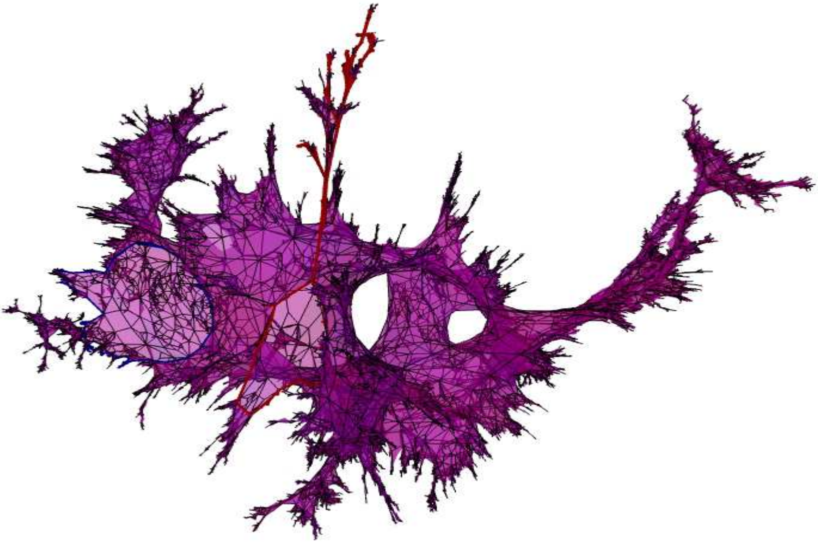
Theorem

The Brownian surface $M_\sigma^{(g)}$ is a.s. homeomorphic to the surface with a boundary $\Sigma_{g,p}^\partial$, where g and p are the integers from the previous slide.

Theorem

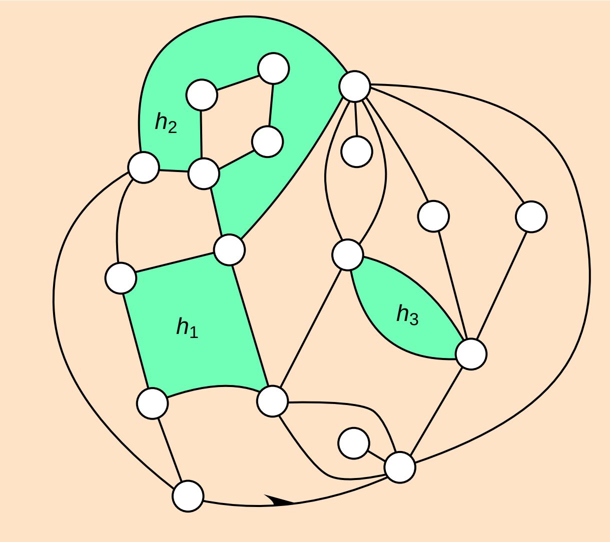
A.s., the Brownian surface $M_\sigma^{(g)}$ has Hausdorff dimension 4 and every connected component of its boundary has Hausdorff dimension 2.

Computer simulations



Simulation of a uniform quadrangulation from $\mathcal{Q}_{30\,000,(100,80)}$ *in genus* 2.

(slight modification of) Bouttier–Di Francesco–Guitter



- Start with a quadrangulation.

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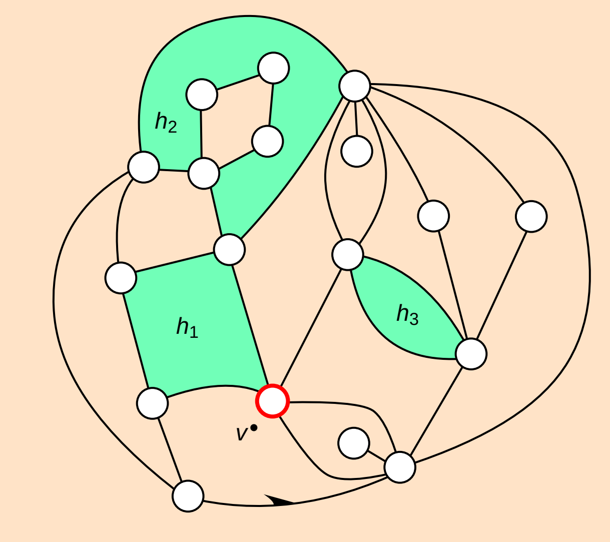

- Start with a quadrangulation.
- Pick a vertex $v^\bullet$.

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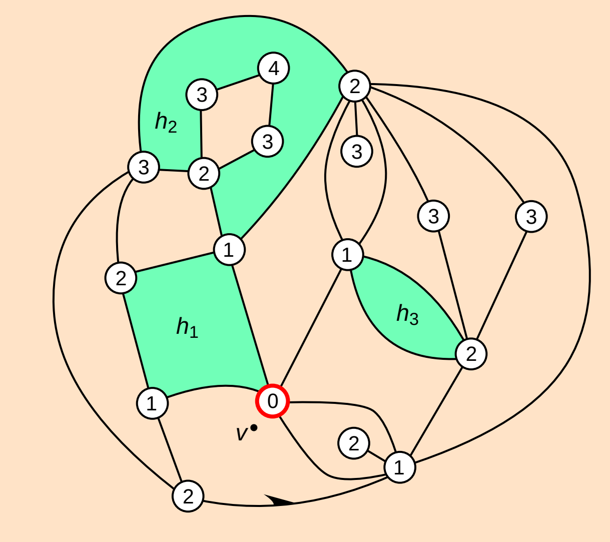

- Start with a quadrangulation.
- Pick a vertex $v^\bullet$.
- Label the vertices with their distance to $v^\bullet$.

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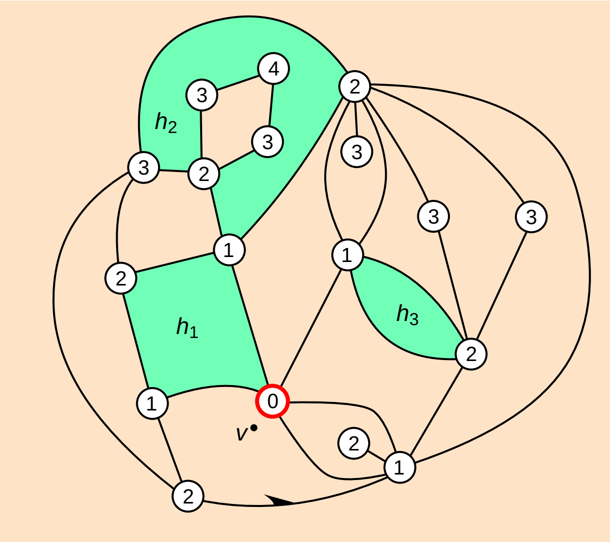

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- Apply the following rule.

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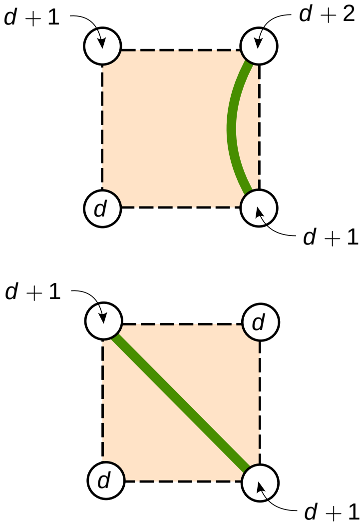

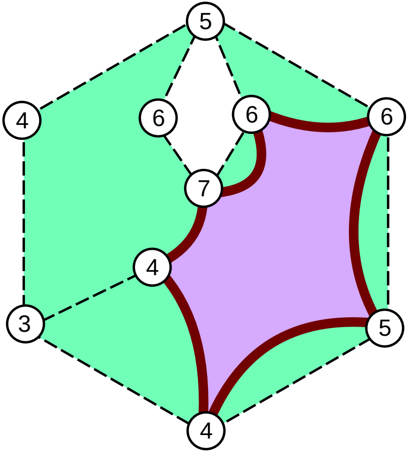

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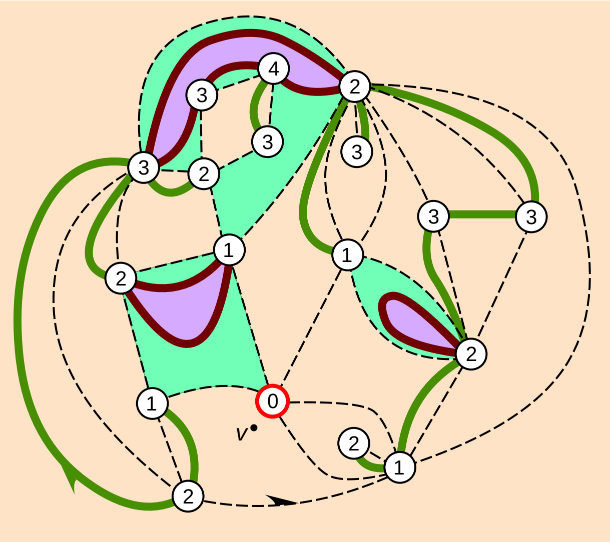

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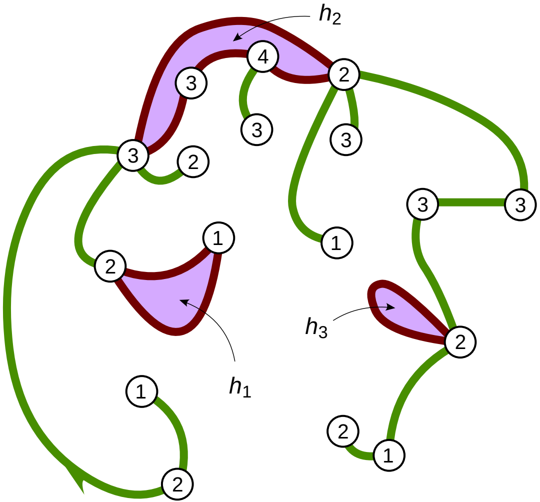
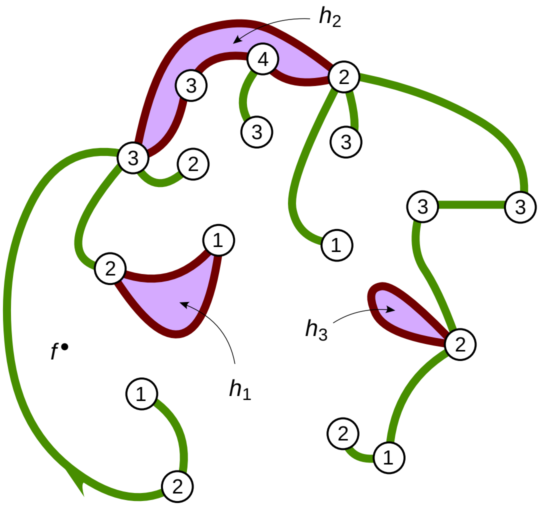


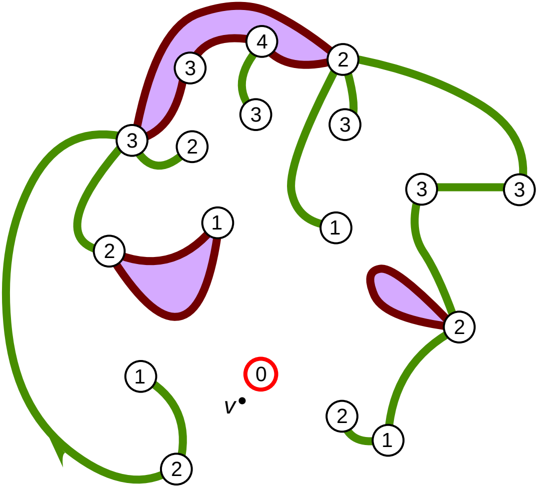
- Start with a quadrangulation.
- Pick a vertex $v^\bullet$.
- Label the vertices with their distance to $v^\bullet$.
- Apply the following rule.
- Remove the initial edges and $v^\bullet$.

Converse construction



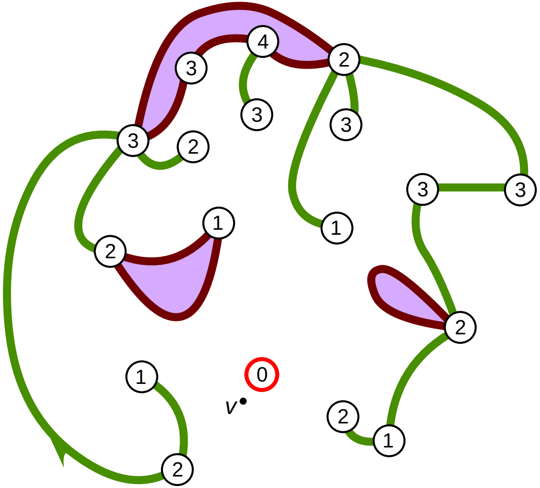
- Take an encoding labeled map.

Converse construction



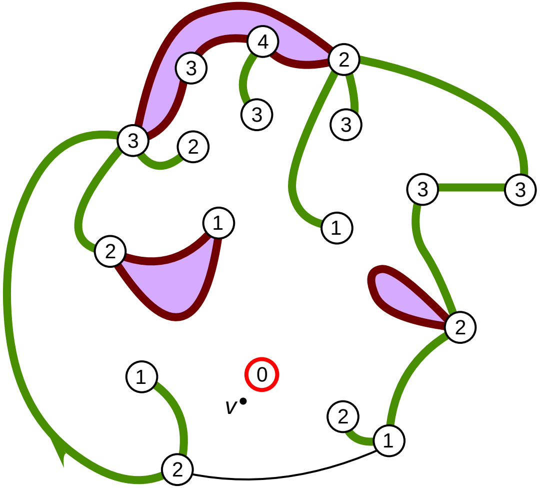
- Take an encoding labeled map.
- Add a vertex $v^\bullet$ inside the internal face $f^\bullet$.

Converse construction



- Take an encoding labeled map.
- Add a vertex $v^\bullet$ inside the internal face $f^\bullet$.
- Link every corner of $f^\bullet$ to the first subsequent corner having a strictly smaller label.

Converse construction



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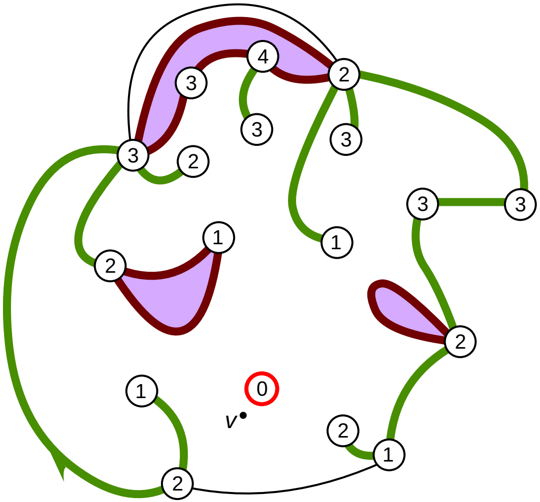

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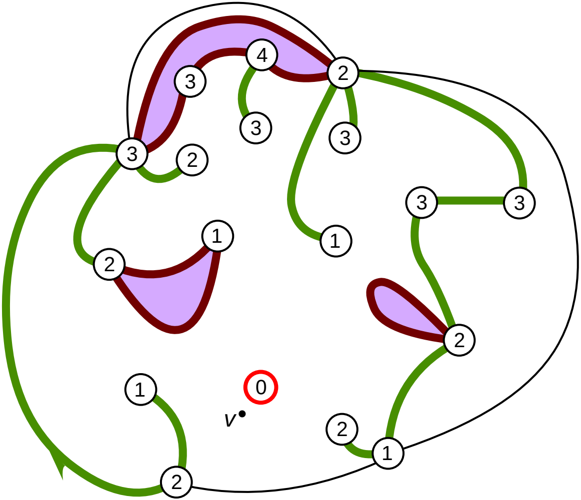

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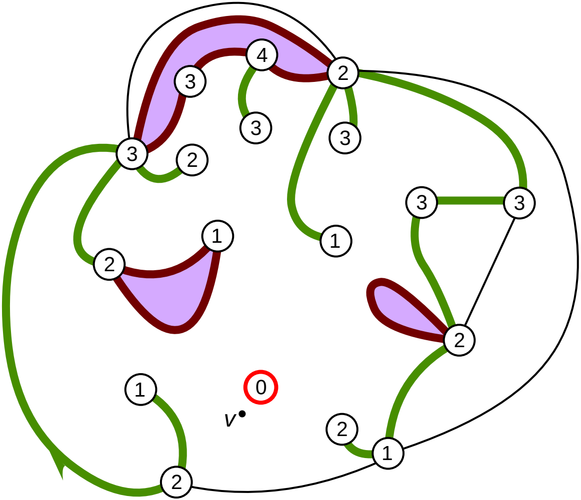

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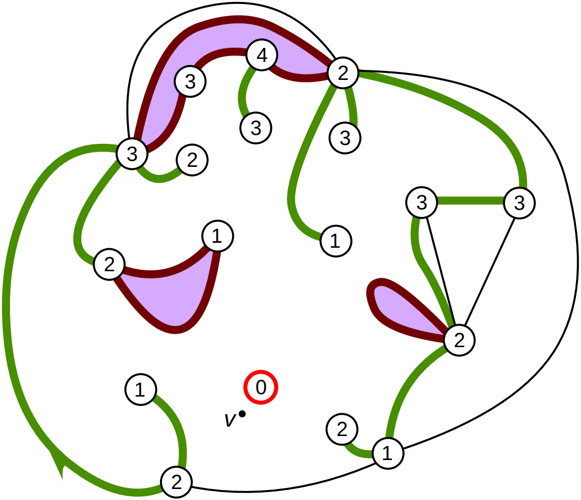

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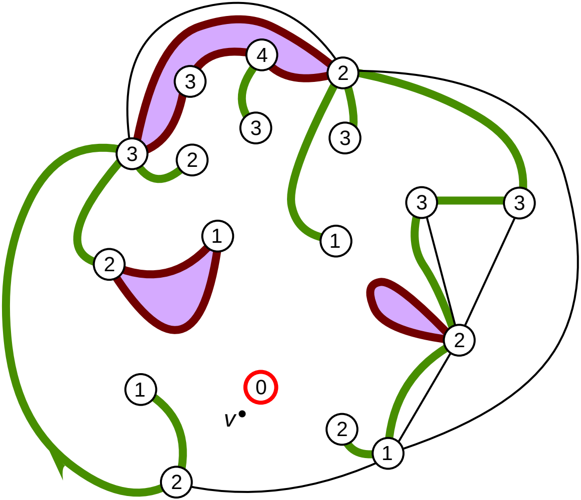

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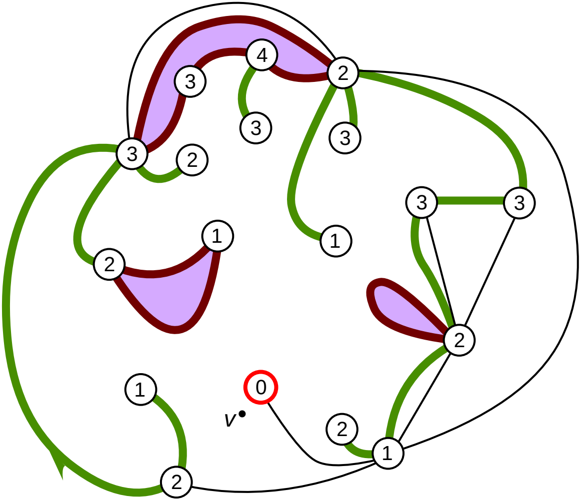

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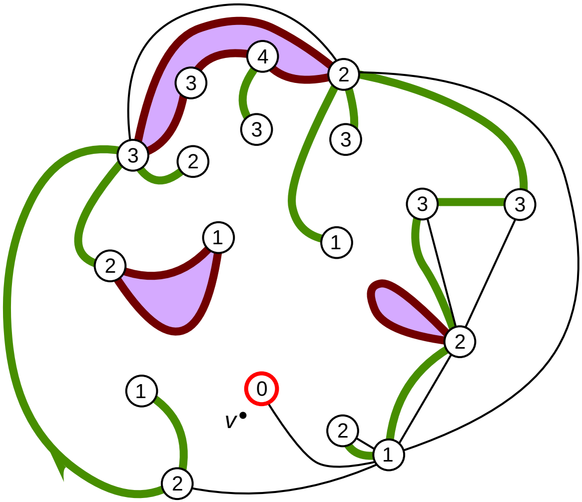

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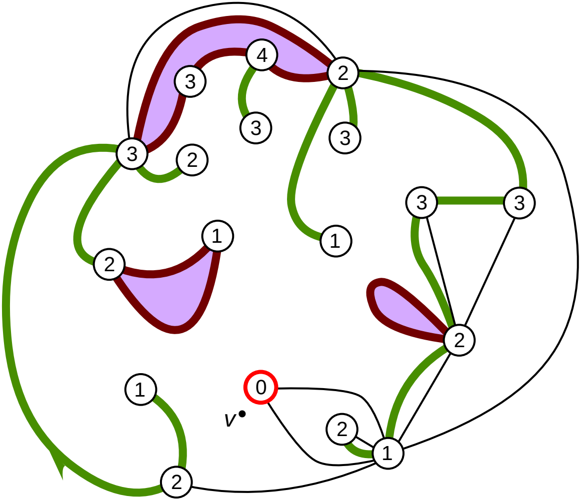

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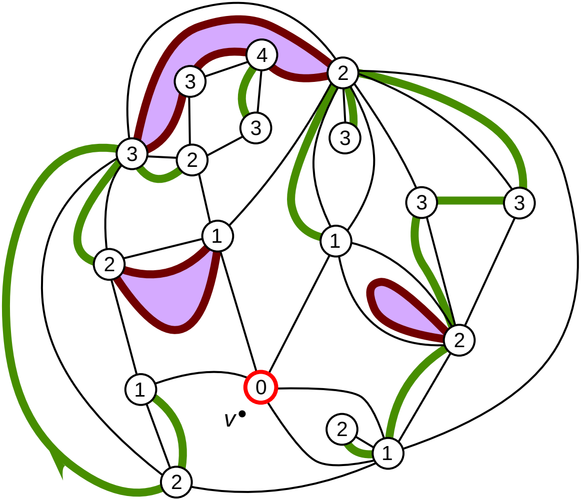

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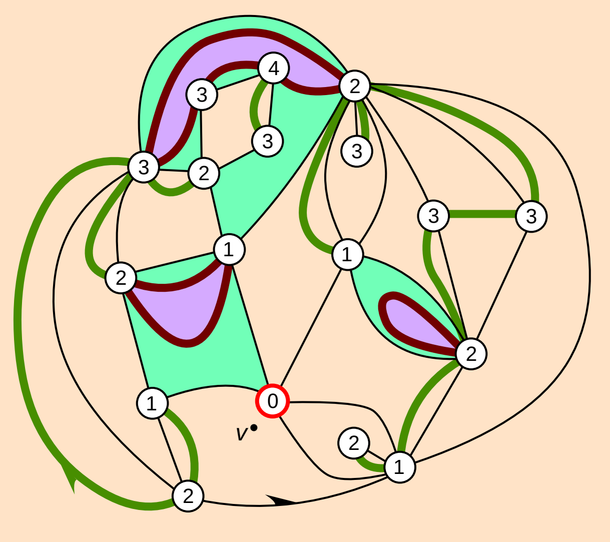

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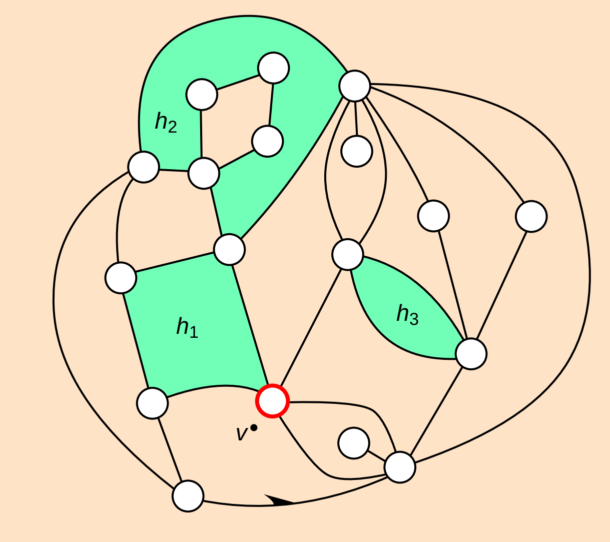

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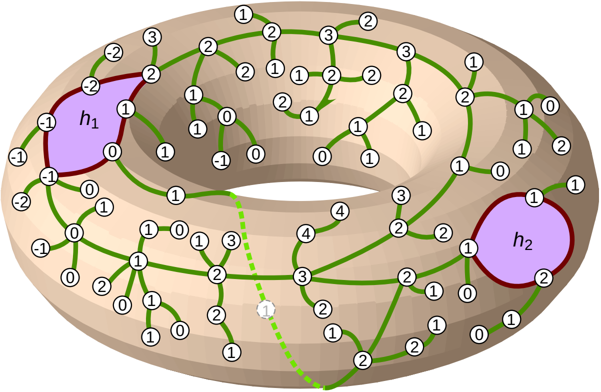
- Add a vertex $v^\bullet$ inside the internal face $f^\bullet$.
- Link every corner of $f^\bullet$ to the first subsequent corner having a strictly smaller label.
- Remove the initial edges.

Decomposition into scheme, bridges and forests



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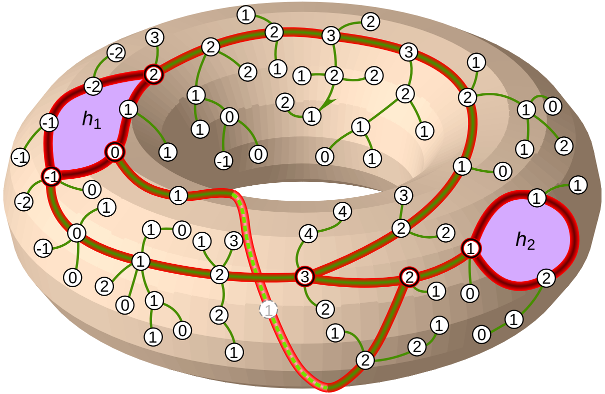

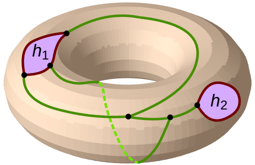

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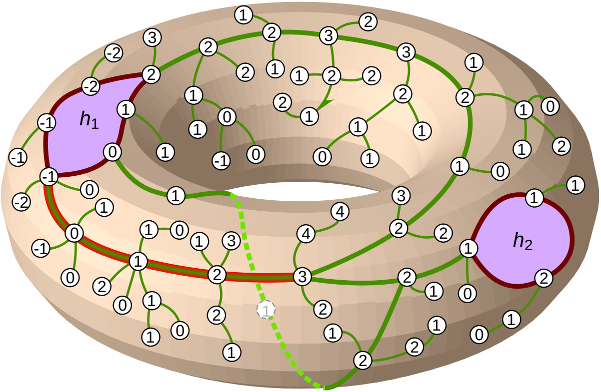
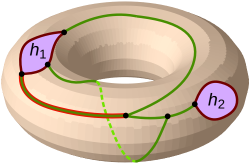
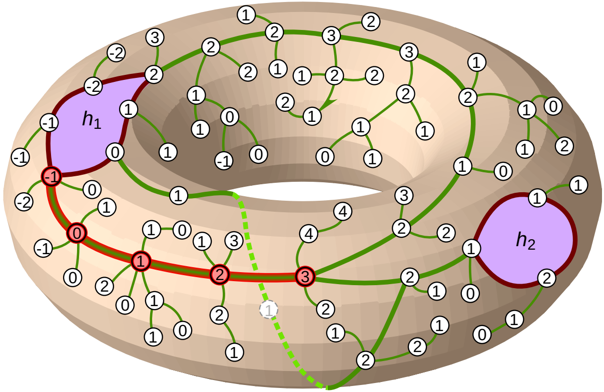


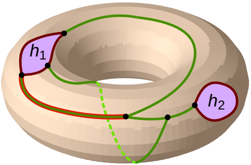


scheme

With each edge of the scheme, we associate:

Decomposition into scheme, bridges and forests





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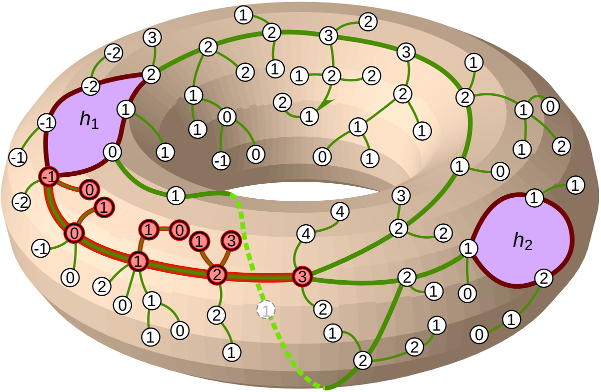
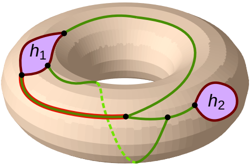
Decomposition into scheme, bridges and forests





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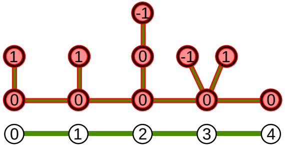
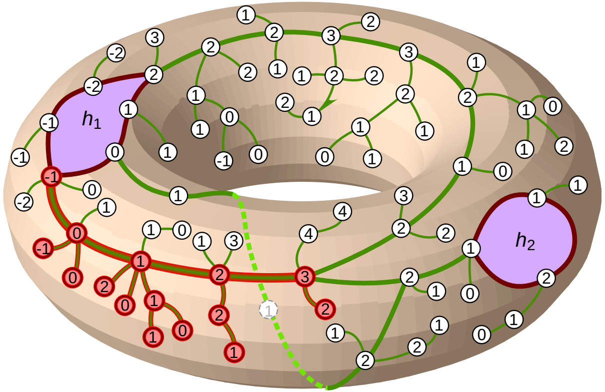
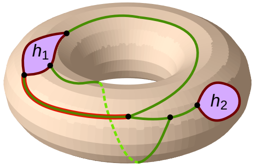


With each edge of the scheme, we associate:

- a Motzkin bridge
- one or two well-labeled forests

Decomposition into scheme, bridges and forests





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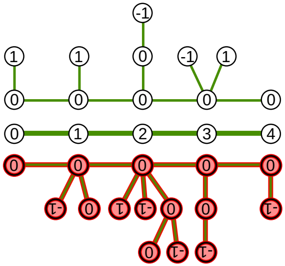

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Schemes

Definition

A scheme with p holes is a genus g map with $p+1$ faces denoted by $h_1, \dots, h_p, f^\bullet$, without vertices of degree ≤ 2 and such that, for every i, h_i has a simple boundary and is not adjacent to any h_j.

Definition

A scheme is dominant if all its vertices have degree 3.

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Definition

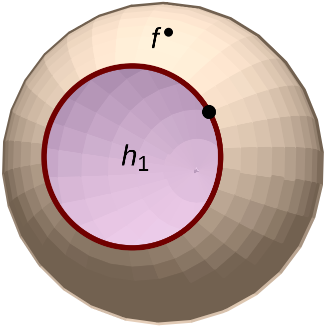
A scheme is dominant if all its vertices have degree 3.

Remark

In reality, the schemes are rooted and the root should satisfy some technical properties. In order to simplify the presentation, we will omit this detail and consider unrooted schemes. Except in the case of the disk, this will not cause any issues.

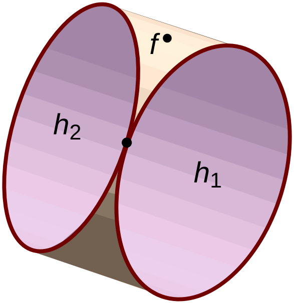
Example 1: the disk ($g = 0$, $p = 1$)

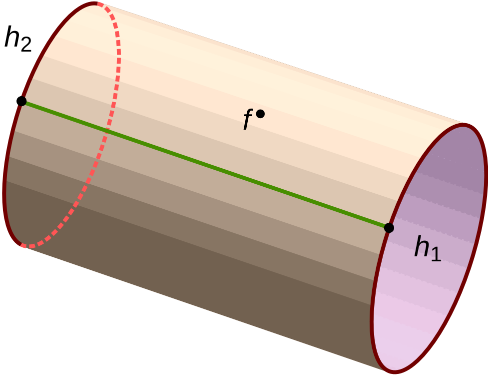
This is a somehow degenerate case: one vertex is allowed to have degree 2 in order for the previous definition not to be void.



The only scheme

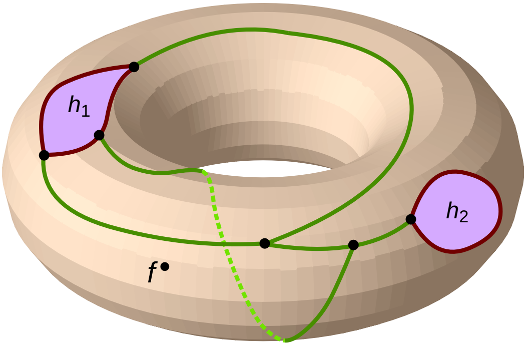
Example 2: the cylinder ($g = 0$, $p = 2$)





The two schemes. The one on the right is dominant.

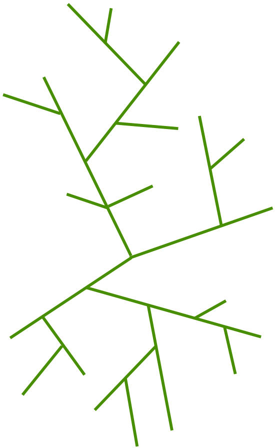
Example 3: surface of genus 1 with 2 holes



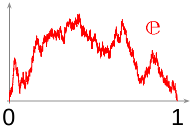
A dominant scheme

Construction $((g, p) = (0, 0))$

Recall how the Brownian map is constructed.



- Consider the CRT $\mathcal{T}_{\mathbb{e}}$, that is, the random real tree encoded by the normalized Brownian excursion.



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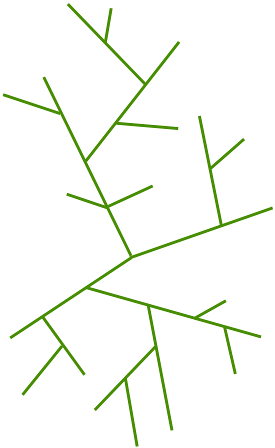

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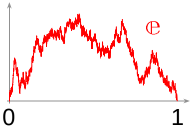

- Put Brownian labels Z on $\mathcal{T}_{\mathbb{e}}$.

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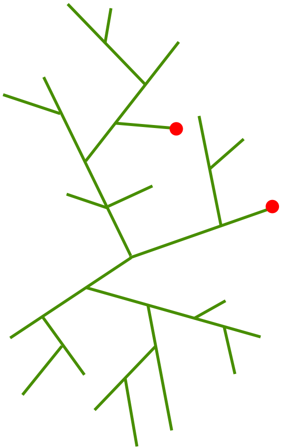

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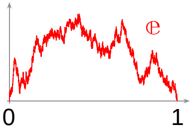

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- Identify the points a and b whenever $Z_a = Z_b = \min_{[a,b]} Z$.

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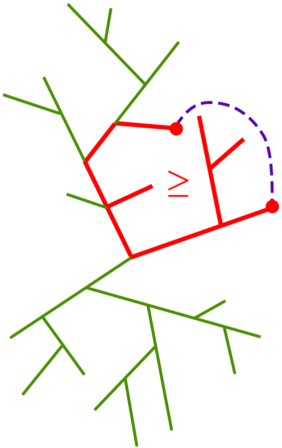

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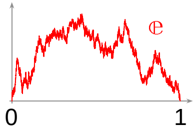

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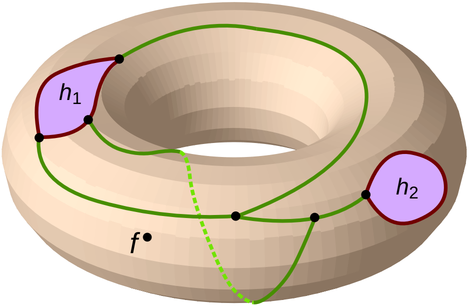
Construction $((g, p) \neq (0, 0))$

Any Brownian surface may be constructed as follows.



- Start with the proper analog to the CRT, denoted by $\mathscr{M}$: it is a dominant scheme with a *Brownian forest* grafted on every half-edge (except inside the holes).

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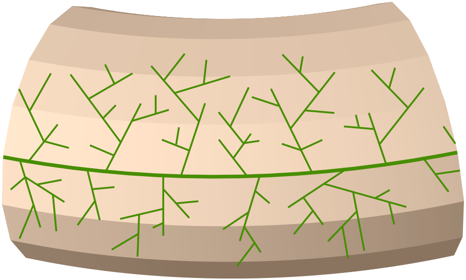

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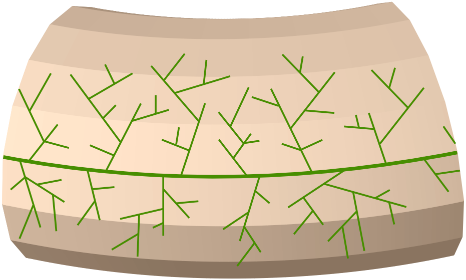

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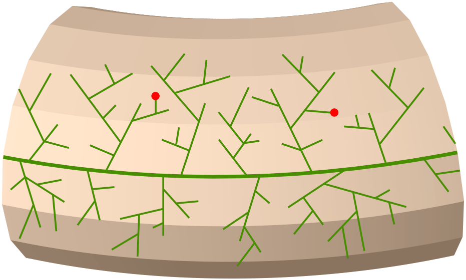

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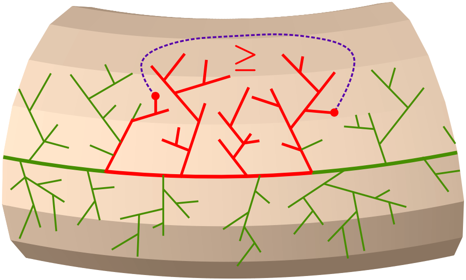

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Important properties of $\mathscr{M}$

Similarly to the CRT, $\mathscr{M}$ may be seen as a quotient of $[0,1]$; informally, this means that there is a natural way to define the "contour" of the unique "internal face" of $\mathscr{M}$.

The corresponding Brownian surface $\left(\mathfrak{q}_\infty^\sigma, d_\infty^\sigma\right)$ may be seen either as a quotient of $\mathscr{M}$ or as a quotient of $[0,1]$.

For $s \in [0,1]$, we will denote by $\mathscr{M}(s)$ and $\mathfrak{q}_\infty^\sigma(s)$ the corresponding points in the quotients.

Lemma

A.s., the labeling function $Z : [0,1] \to \mathbb{R}$ reaches its minimum only once.

We set $s^\bullet := \operatorname{argmin} Z$ and we denote by $\rho^\bullet := \mathfrak{q}_\infty^\sigma(s^\bullet)$ the corresponding point in the surface.

Geodesic spaces

Definition

In a compact metric space $(\mathcal{X}, \delta)$, a geodesic from $x \in \mathcal{X}$ to $y \in \mathcal{X}$ is a continuous path $\wp : [0, \delta(x, y)] \to \mathcal{X}$ such that $\wp(0) = x$, $\wp(\delta(x, y)) = y$ and $\delta(\wp(s), \wp(t)) = |t - s|$ for all $s, t \in [0, \delta(x, y)]$.

Definition

A geodesic space is a compact metric space in which every pair of points is connected by (at least) one geodesic.

Proposition

Every Brownian surface is a.s. a geodesic space.

Simple geodesics

In a discrete map, the simple geodesic starting from a corner is obtained by following the subsequent edges of the previous bijection. The continuous analog is the following:

Definition

The simple geodesic of index $s \in [0,1]$ is the path Φ_s defined by

$$\Phi_s(w) := \mathfrak{q}_\infty^\sigma \left(\inf \left\{ r \ : \ \inf_{[s \to r]} Z = Z_{s^\bullet} + w \right\} \right), \qquad 0 \le w \le d_\infty^\sigma(s^\bullet, s),$$

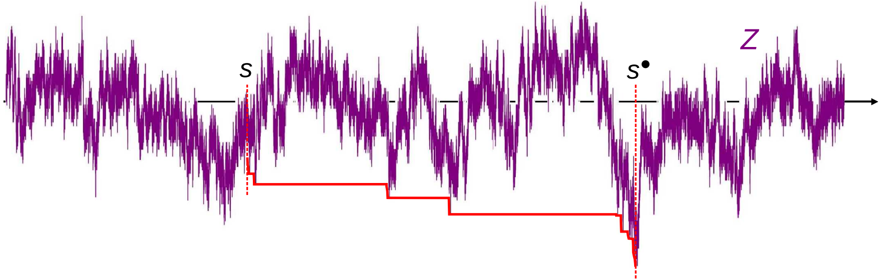
where

$$[s \to t] := \begin{cases} [s,t] & \text{if } s \le t, \\ [s,1] \cup [0,t] & \text{if } t < s. \end{cases}$$

Remark

$d_\infty^\sigma(s^\bullet, s) = Z_s - Z_{s^\bullet}$.

Simple geodesics



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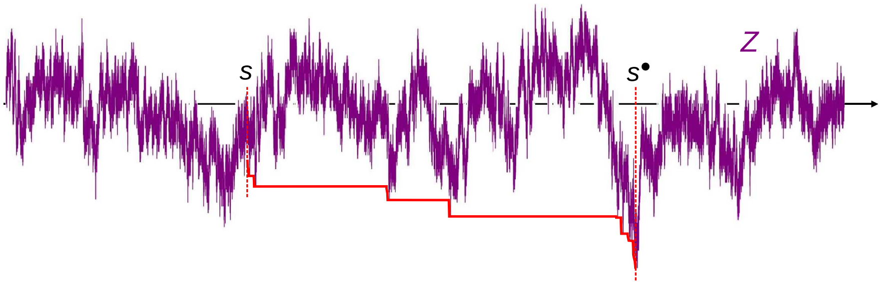

Proposition (Le Gall '10)

Simple geodesics are geodesics from $\rho^\bullet$.

Simple geodesics

Theorem (Le Gall '10 [$(g, p) = (0, 0)$], B. '14)

A.s., all the geodesics from $\rho^\bullet$ are simple geodesics.

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Proposition

A.s., all the points visited by a simple geodesic are leaves (points $a \in \mathscr{M} \setminus \partial\mathscr{M}$ s.t. $\exists!\, s$ for which $a = \mathscr{M}(s)$), except possibly the endpoint.

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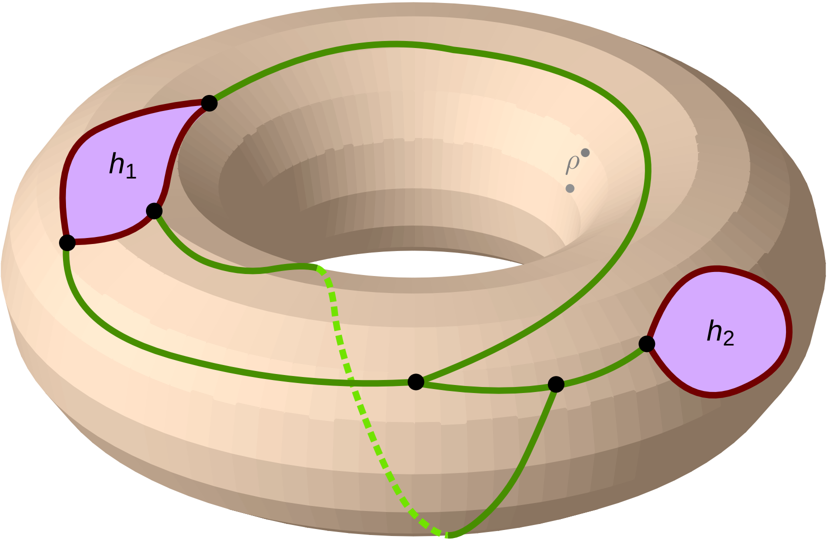
We denote by e_s the half-edge of the scheme “carrying” the forest to which $\mathscr{M}(s)$ belongs.

Proposition

Let s and t be such that $\mathscr{M}(s) = \mathscr{M}(t)$. The path $\Phi_s \bullet \bar{\Phi}_t$ is homotopic to 0 if and only if $e_s = e_t$.

Simple geodesics



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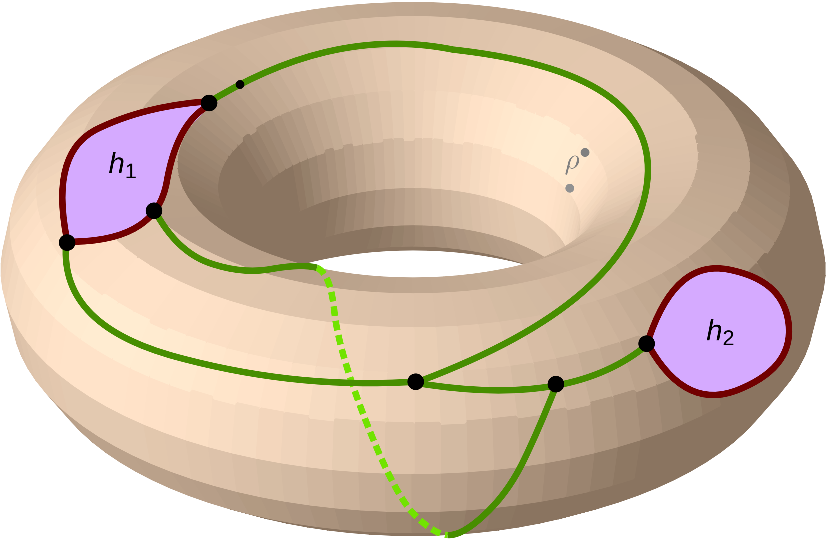

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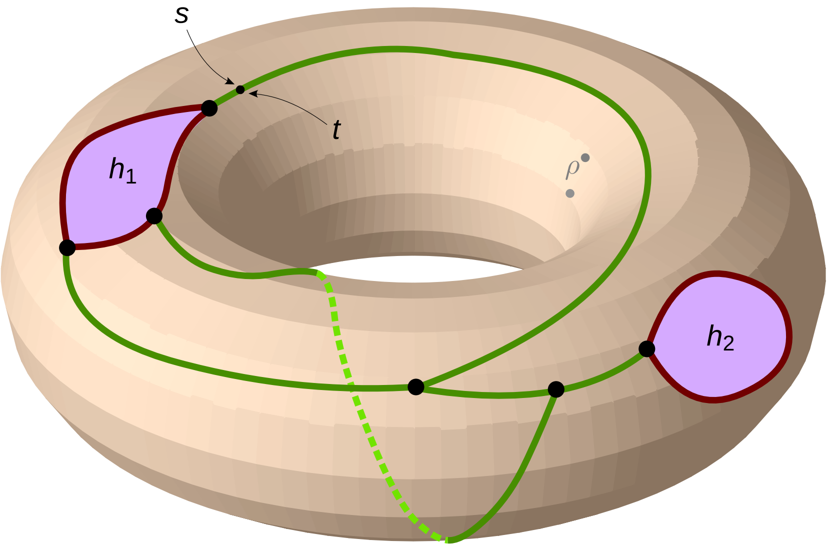

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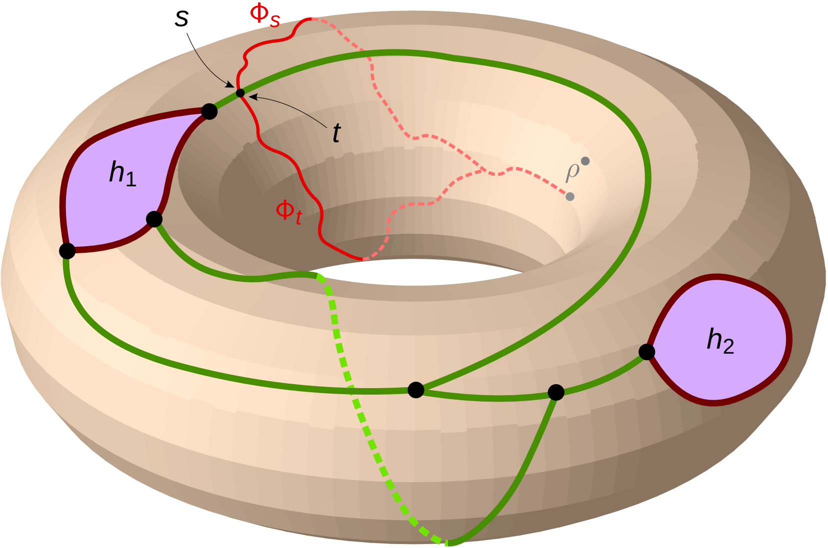

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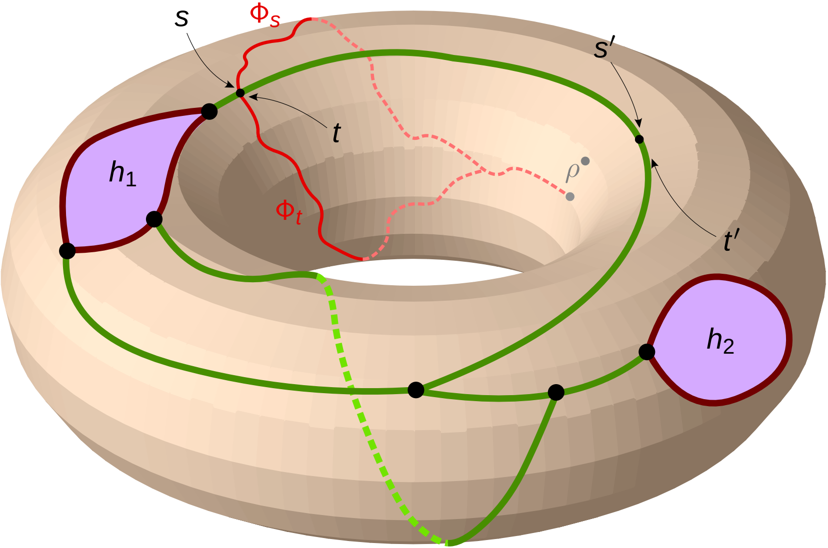

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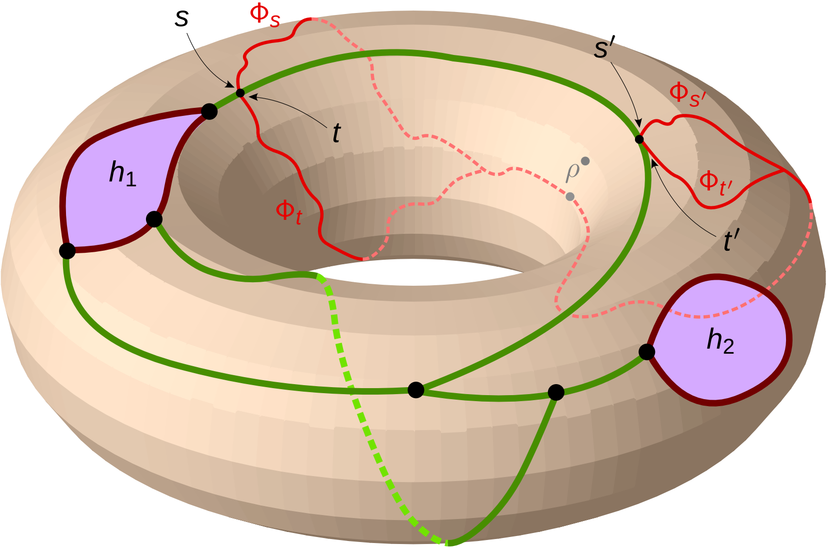

Geodesics concatenations not homotopic to 0

Let $\mathcal{N}(\rho^\bullet, \mathfrak{q}_\infty^\sigma)$ denote the set of points $x \in \mathfrak{q}_\infty^\sigma$ for which there exist at least one pair $\{\wp, \wp'\}$ of geodesics from $\rho^\bullet$ to x such that $\wp \bullet \bar{\wp}'$ is not homotopic to 0.

Proposition

$\mathcal{N}(\rho^\bullet, \mathfrak{q}_\infty^\sigma)$ *corresponds to the union of the edges of the scheme that are not incident to the holes (the green ones).*

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Proposition

$\mathcal{N}(\rho^\bullet, \mathfrak{q}_\infty^\sigma)$ corresponds to the union of the edges of the scheme that are not incident to the holes (the green ones).

Theorem

Except for the sphere and the disk, the Hausdorff dimension of $\mathcal{N}(\rho^\bullet, \mathfrak{q}_\infty^\sigma)$ is a.s. 2.

Number of geodesics

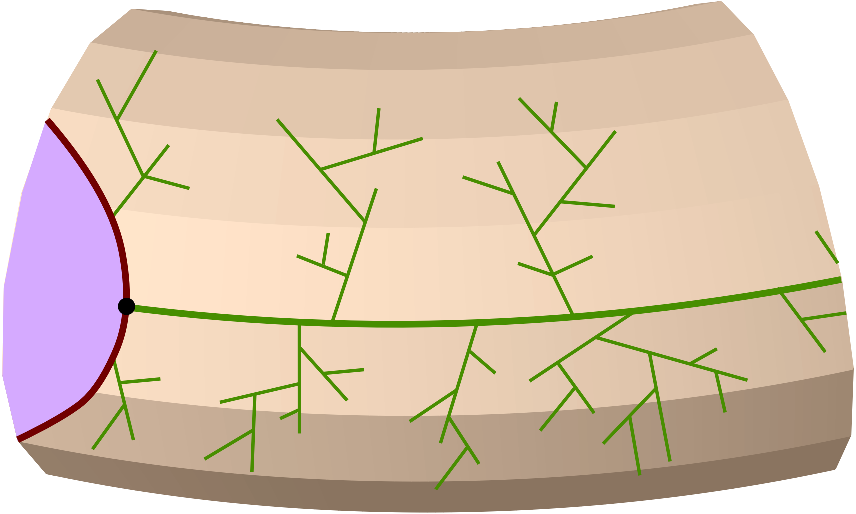
We denote by $\pi : \mathscr{M} \to \mathfrak{q}_\infty^\sigma$ the canonical projection.

Theorem

A.s., for every a $\in \mathscr{M}$, the number of geodesics from $\rho^\bullet$ to $\pi(a)$ is equal to the maximal number of connected components of $\mathscr{M}\setminus\{a\}$ restricted to the neighborhoods of a, minus $\mathbb{1}_{\{\pi(a)\in\partial\mathfrak{q}_\infty^\sigma\}}$.

In particular, this number is typically 1*, at most* 2 *for the boundary points and at most* 3 *for internal points.*

Number of geodesics



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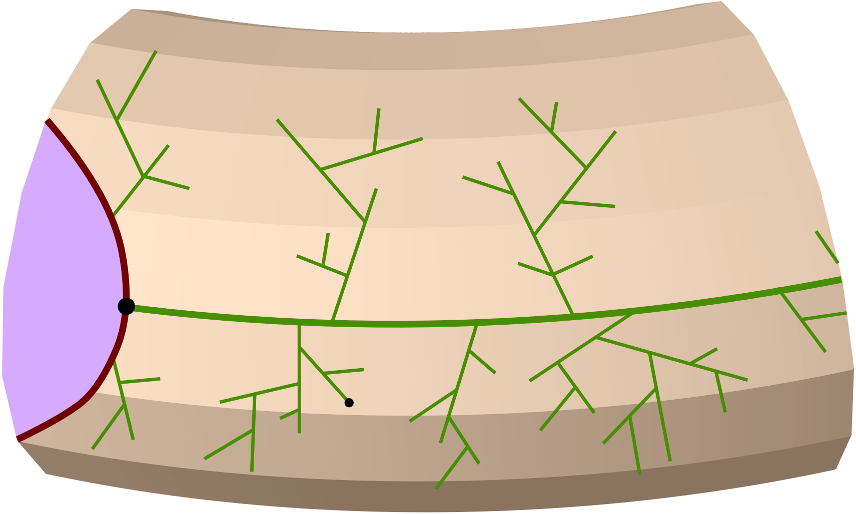

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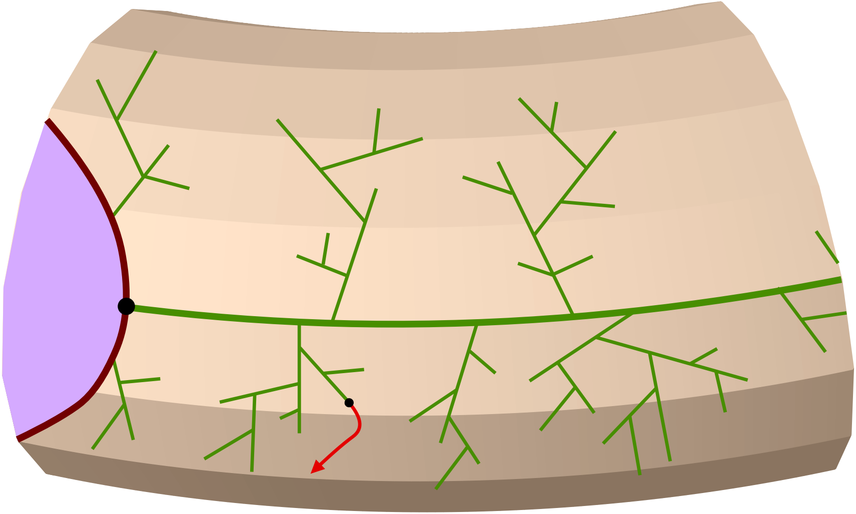

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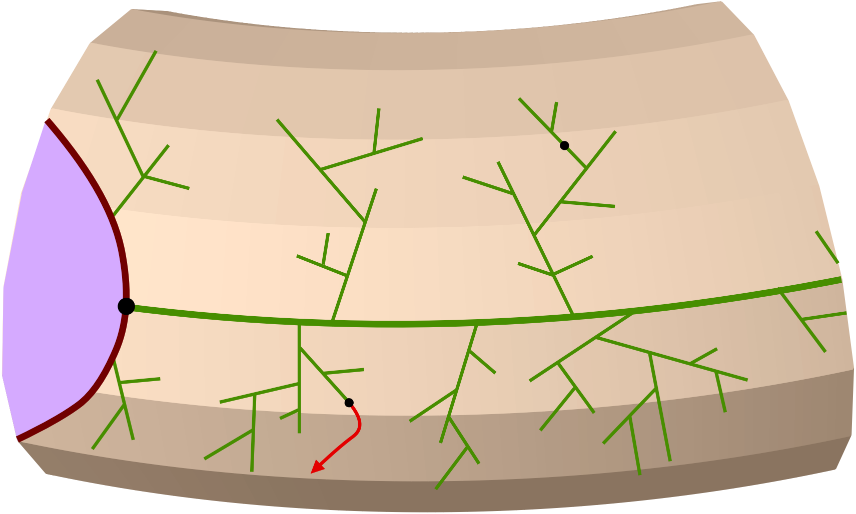

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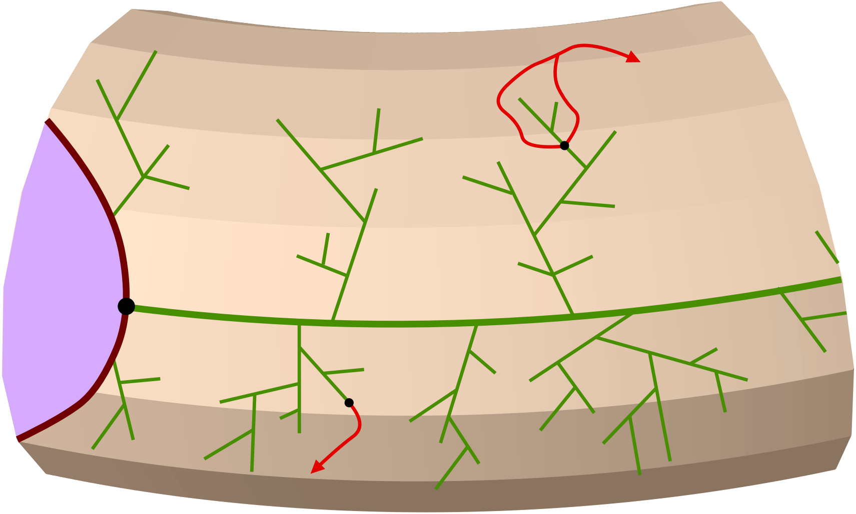

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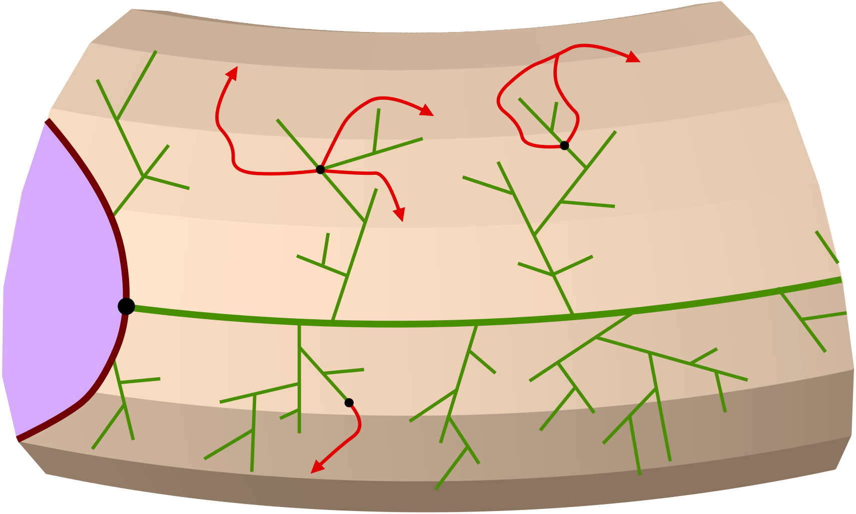

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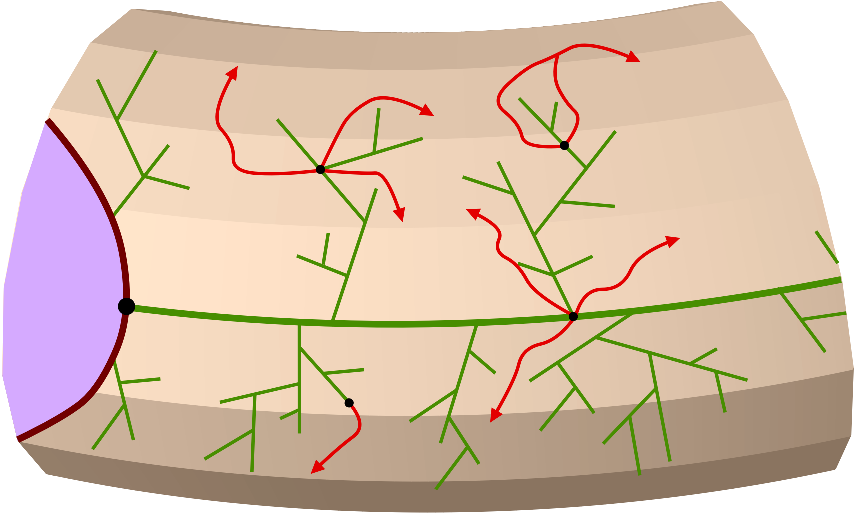

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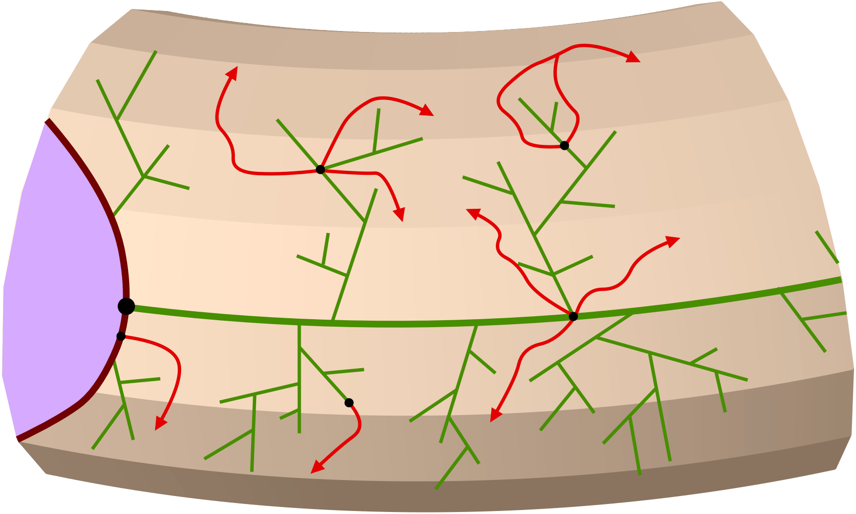

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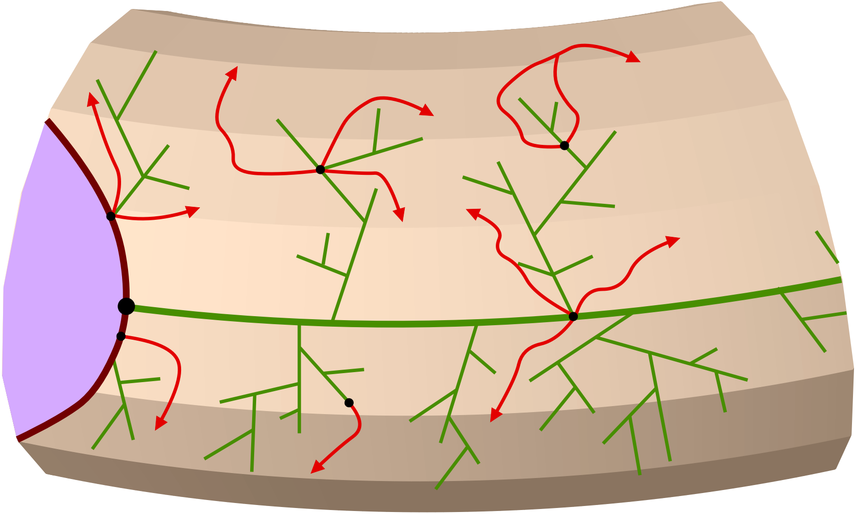


Very peculiar points

Theorem

The set of points reachable by 3 distinct geodesics and for which every pair $\{\wp, \wp'\}$ of geodesics is such that $\wp \bullet \bar{\wp}'$ is not homotopic to 0 is finite: its cardinality H is

- *equal to 0 if $g = 0$ and $p \in \{0, 1\}$;*
- *equal to $4g - 2$ if $g \geq 1$ and $p = 0$;*
- *a random variable whose distribution only depends on g, p and σ and whose support is $\{0, 1, \dots, 4g + p - 2\}$ otherwise.*

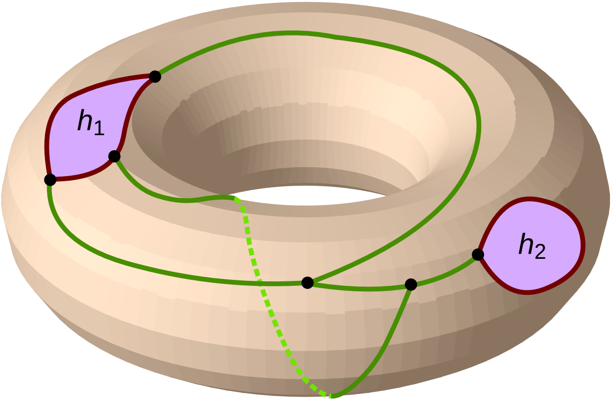
Theorem

Suppose $p \neq 0$. The set $\partial\mathfrak{q}_\infty^\sigma \cap \mathcal{N}(\rho^\bullet, \mathfrak{q}_\infty^\sigma)$ is finite: its cardinality is $4g + 2p - 2 - H$.

Very peculiar points

Proposition

The previous points correspond to the vertices of the scheme. There is a dichotomy depending on whether they are incident to a hole or not.



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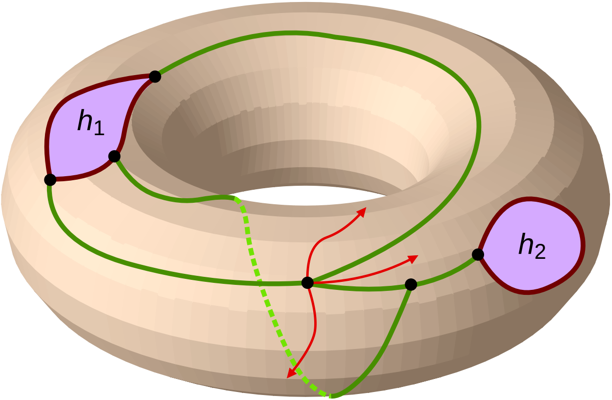

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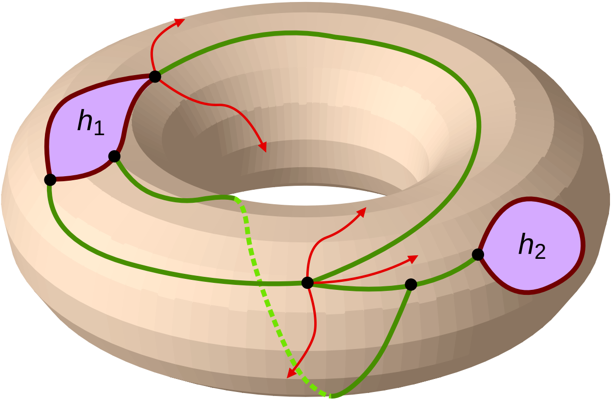


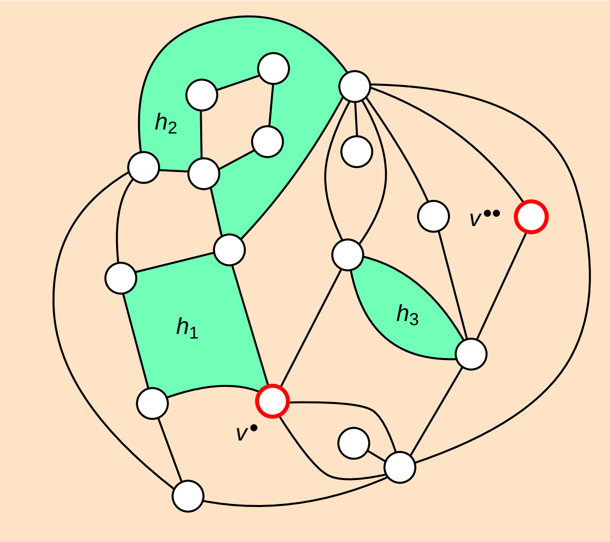
Confluence of geodesics

The confluence property shown by Le Gall for the Brownian map easily translates for any Brownian surface.

Proposition (Le Gall '10)

A.s., for every $\varepsilon > 0$, there exists $\eta \in (0, \varepsilon)$ such that all the geodesics from $\rho^\bullet$ to points outside of the ball of radius ε centered at $\rho^\bullet$ share a common initial part of length η.

(slight modification of) Miermont's 2-point bijection



- Start with a quadrangulation, two vertices $v^{\bullet}$, $v^{\bullet\bullet}$ and $1 \leq \lambda \leq d - 1$, where $d := d(v^{\bullet}, v^{\bullet\bullet})$.

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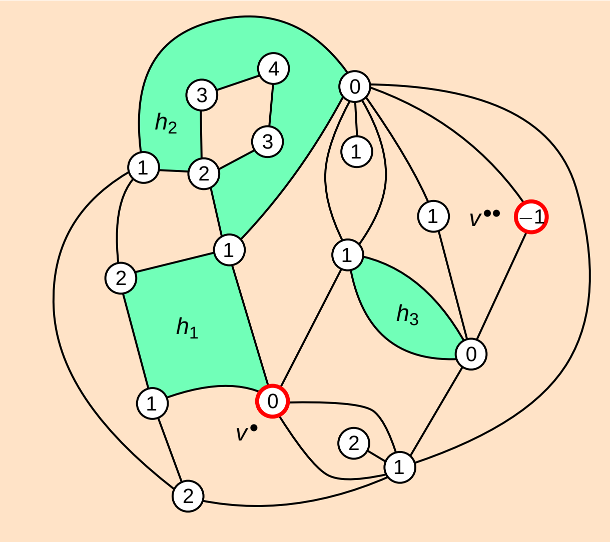

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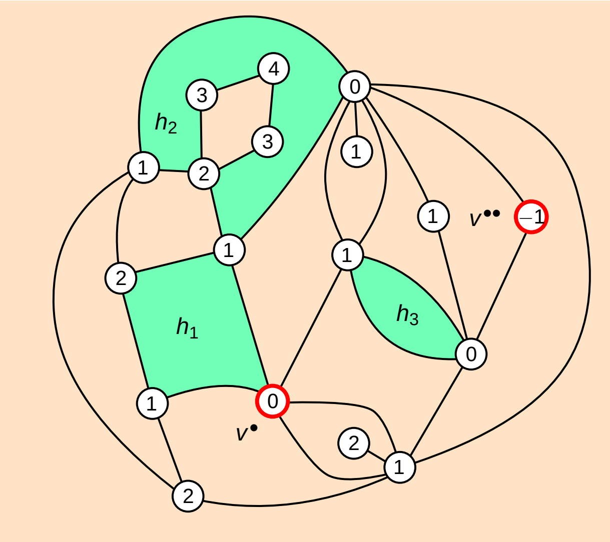

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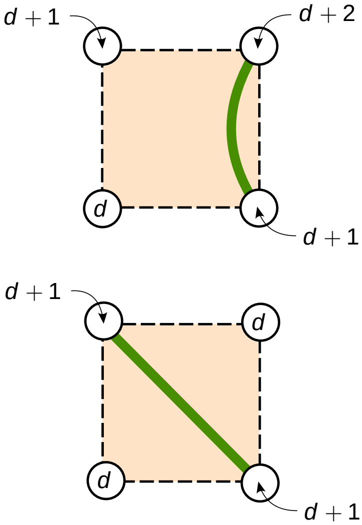

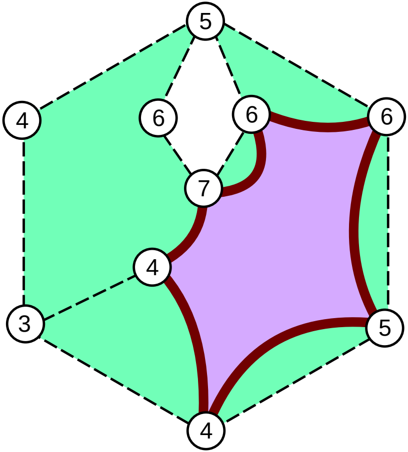

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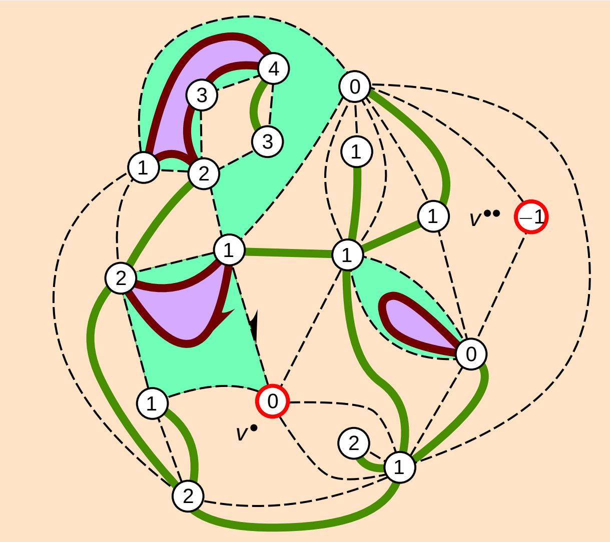

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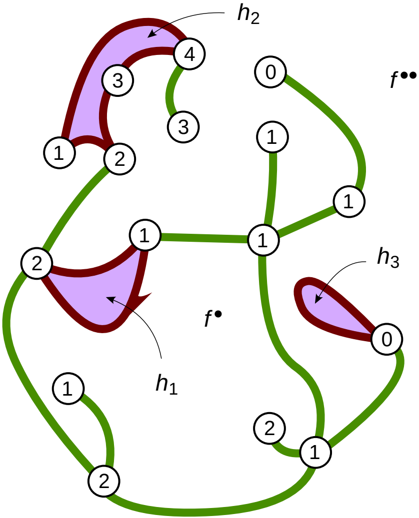

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- Label each v by $d(v^{\bullet}, v) \wedge d(v^{\bullet\bullet}, v) + 2\lambda - d$.
- Apply the following rule.
- Remove the initial edges, $v^{\bullet}$, $v^{\bullet\bullet}$.

Uniqueness of typical geodesics

Proposition (inspired from Miermont '09)

Let S be uniformly distributed over $[0,1]$ and independent of $(\mathfrak{q}^\sigma_\infty, d^\sigma_\infty)$. Then, a.s., there is only one geodesic from $\rho^\bullet$ to $X := \mathfrak{q}^\sigma_\infty(S)$.

Rough idea. We consider an r.v. U uniform on $[0,1]$ and we show that

$$\{y \in \mathfrak{q}^\sigma_\infty \,:\, d^\sigma_\infty(\rho^\bullet, y) = U\, d^\sigma_\infty(\rho^\bullet, X) \text{ et } d^\sigma_\infty(y, X) = (1-U)\, d^\sigma_\infty(\rho^\bullet, X)\}$$

is a.s. a singleton.

The points of this set correspond to global minimums of the labeling function on the interface between the two “faces” of the scaling limit of the map obtained by performing Miermont's 2-point bijection with the proper choice of parameters.

As these labels are essentially Brownian, there may only be one global minimum. □

Uniqueness of typical geodesics

Consequence

Let $(s_i)_{i\geq 0}$ be a sequence of i.i.d. uniform r.v. on $[0,1]$.

We set $a_i := \mathscr{M}(s_i)$ and $x_i := \mathfrak{q}_\infty^\sigma(s_i)$.

Then, a.s., for every i, Φ_{s_i} is the only geodesic from $\rho^\bullet$ to x_i.

Moreover, a.s., $\{s_i : i \geq 0\}$ is a dense subset of $[0,1]$. Up to discarding a zero-probability event, we suppose that both previous properties hold.

Uniqueness of typical geodesics

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Let $(s_i)_{i\geq 0}$ be a sequence of i.i.d. uniform r.v. on $[0,1]$.
We set $a_i := \mathscr{M}(s_i)$ and $x_i := \mathfrak{q}^{\sigma}_{\infty}(s_i)$.
Then, a.s., for every i, Φ_{s_i} is the only geodesic from $\rho^\bullet$ to x_i.

Moreover, a.s., $\{s_i \,:\, i \geq 0\}$ is a dense subset of $[0,1]$. Up to discarding a zero-probability event, we suppose that both previous properties hold.

We will limit ourselves to the case of leaves, the general case being hardly more complicated. More precisely, we want the following.

Goal

Let s be such that $a := \mathscr{M}(s)$ is a leaf. We want to show that Φ_s is the only geodesic from $\rho^\bullet$ to $x := \mathfrak{q}^{\sigma}_{\infty}(s)$.

Geodesics do not climb down trees

For b and c in a subtree of $\mathscr{M}$, we let $[[b, c]]$ denote the unique range of the injective paths linking b to c in the subtree.

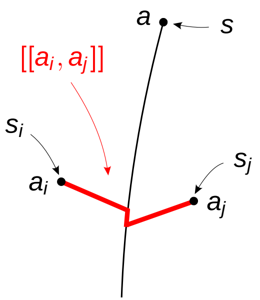
Lemma (1)

Let $\varepsilon > 0$ be such that $\{\mathscr{M}(t) : s - \varepsilon \le t \le s + \varepsilon\}$ does not intersect the scheme, except maybe at one point.
There exist i and j satisfying

$$s - \varepsilon < s_i < s < s_j < s + \varepsilon$$

and, for all $b \in [[a_i, a_j]]$,

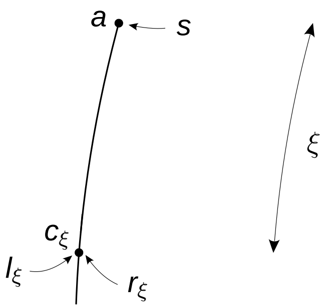
$$d_\infty^\sigma(\rho^\bullet, a) < d_\infty^\sigma(\rho^\bullet, b) + d_\infty^\sigma(b, a).$$



Geodesics do not climb down trees

Idea. We argue by contradiction.

For a small fixed ξ, we let c_ξ be the point of the tree at distance ξ from a in the tree.



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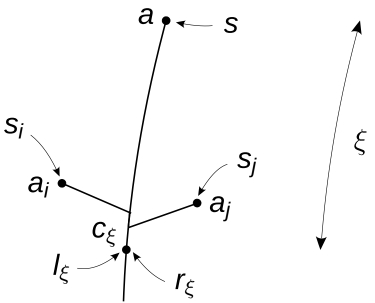
For a small fixed ξ, we let c_ξ be the point of the tree at distance ξ from a in the tree.

For any small η, we can find
$l_\xi \leq s_i < l_\xi + \eta$ and $r_\xi - \eta < s_j \leq r_\xi$,



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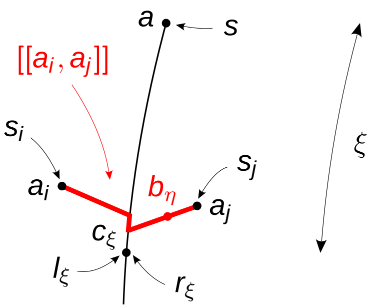
For any small η, we can find $l_\xi \leq s_i < l_\xi + \eta$ and $r_\xi - \eta < s_j \leq r_\xi$, and hence some $b_\eta \in [[a_i, a_j]]$ s.t. $d_\infty^\sigma(\rho^\bullet, a) = d_\infty^\sigma(\rho^\bullet, b_\eta) + d_\infty^\sigma(b_\eta, a)$.



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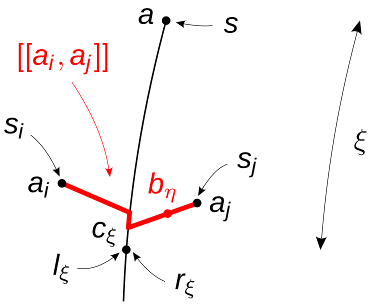

As $\eta \to 0$, up to extraction, "$b_\eta \to c_\xi$" and we obtain

$$d_\infty^\sigma(c_\xi, a) = d_\infty^\sigma(\rho^\bullet, a) - d_\infty^\sigma(\rho^\bullet, c_\xi) = Z_a - Z_{c_\xi}.$$

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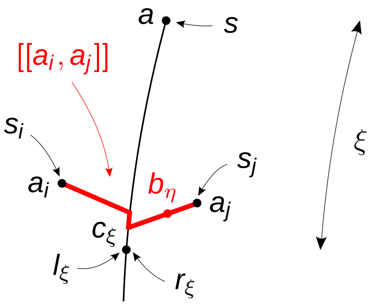

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$$d_\infty^\sigma(c_\xi, a) = d_\infty^\sigma(\rho^\bullet, a) - d_\infty^\sigma(\rho^\bullet, c_\xi) = Z_a - Z_{c_\xi}.$$

By the "cactus bound," this is $\geq Z_a + Z_{c_\xi} - 2\min_{[[c_\xi, a]]} Z$, so that $Z_{c_\xi} = \min_{[[c_\xi, a]]} Z$. As a result, $\xi \mapsto Z_{c_\xi}$ is nonincreasing. □

How to get out of a tree?

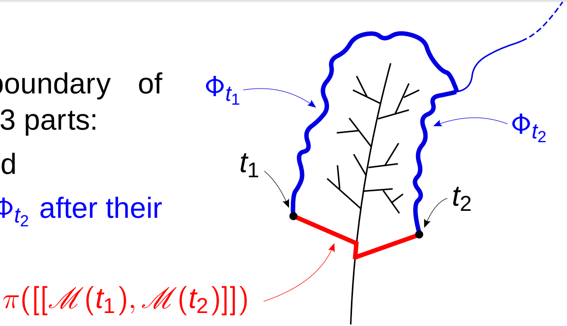
Lemma (2)

Let τ be a tree of $\mathscr{M}$ and $t_1 \le t_2$ s.t. $\{\mathscr{M}(t) : t_1 \le t \le t_2\} \subseteq \tau$. Let $w' := \inf\{w : \Phi_{t_1}(w) \neq \Phi_{t_2}(w)\}$ be the instant at which Φ_{t_1} and Φ_{t_2} split up. The topological boundary of $\mathfrak{q}^\sigma_\infty([t_1, t_2])$ is

$$\pi\big([[\mathscr{M}(t_1), \mathscr{M}(t_2)]]\big) \cup \Phi_{t_1}\big([w', d^\sigma_\infty(s^\bullet, t_1)]\big) \cup \Phi_{t_2}\big([w', d^\sigma_\infty(s^\bullet, t_2)]\big).$$

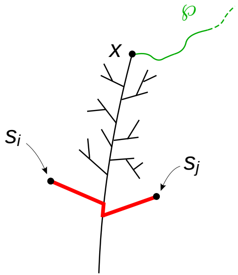
In other words, the boundary of $\mathfrak{q}^\sigma_\infty([t_1, t_2])$ is composed of 3 parts:

- $\pi([[\mathscr{M}(t_1), \mathscr{M}(t_2)]])$ and
- the ranges of Φ_{t_1} and Φ_{t_2} after their separation.



Φ_s is the only geodesic from $\rho^\bullet$ to x

Proof. Let $\wp : [0, d_\infty^\sigma(\rho^\bullet, x)] \to \mathfrak{q}_\infty^\sigma$ be a geodesc from $\rho^\bullet$ to x and $\varepsilon > 0$ be small. We choose s_i and s_j satisfying the conditions of Lemma (1).



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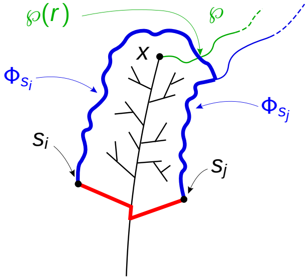
We set $r := \sup\{w \; : \; \wp(w) \notin \mathfrak{q}_\infty^\sigma([s_i, s_j])\}$, so that $\wp(r)$ belongs to the boundary of $\mathfrak{q}_\infty^\sigma([s_i, s_j])$. As, by definition, $\wp(r) \notin \pi([[a_i, a_j]])$, Lemma (2) yields that $\wp$ meets Φ_{s_i} or Φ_{s_j}.



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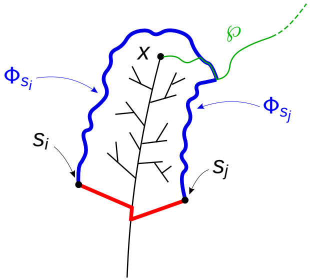
As Φ_{s_i} and Φ_{s_j} are the only geodesics from $\rho^\bullet$ to x_i and x_j, we deduce that $\wp$ coincides with Φ_{s_i} and Φ_{s_j} on the interval where they are equal.



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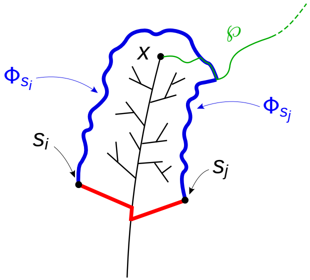
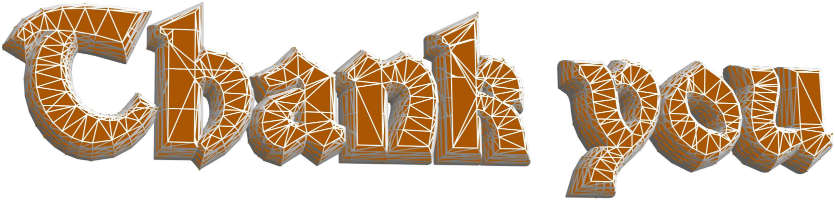
In particular, this is also true for Φ_s and thus $\wp$ coincides with Φ_s up to the instant where Φ_{s_i} and Φ_{s_j} split up.

We conclude by letting $\varepsilon \to 0$. □




Thank you