# Introduction aux cartes aléatoires Soutenance de magistère

Jérémie BETTINELLI

Octobre 2008

## Table des matières

- Définitions
  - Description classique
  - Formule d'Euler
  - Description combinatoire
- Méthodes de comptage par décomposition récursive
- Approches bijectives
  - Bijection de Gilles SCHAEFFER
  - Limite d'échelle



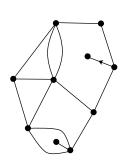
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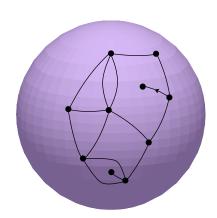
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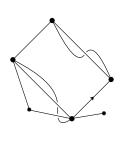


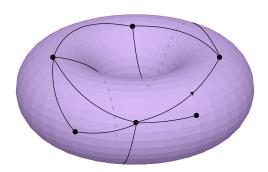
## Cartes, faces, genre, racine





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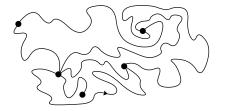




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Description classique Formule d'Euler Description combinatoire

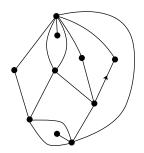
# Cartes isomorphes







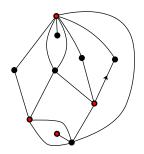
# Quadrangulations





Description classique Formule d'Euler Description combinatoire

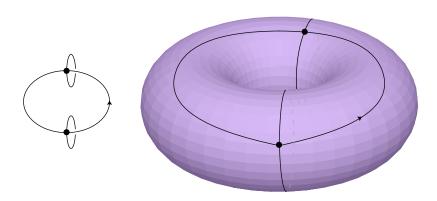
# Cartes biparties





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# Cartes biparties





#### Proposition

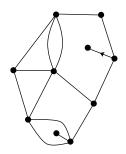


#### Proposition



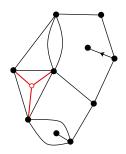


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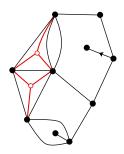




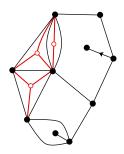
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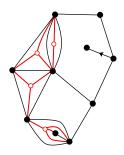
#### Proposition



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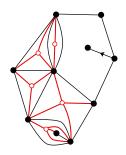


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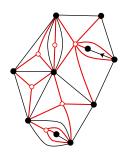




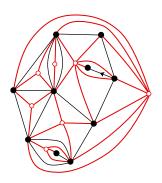
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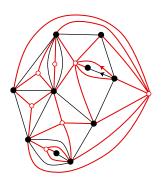
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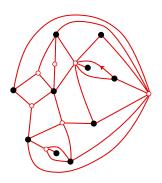
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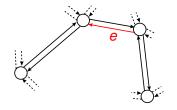
## Formule d'Euler

#### Proposition

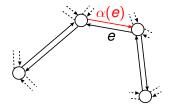
On a

$$|V(m)| - |E(m)| + |F(m)| = 2 - 2g.$$

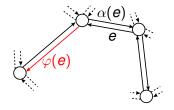




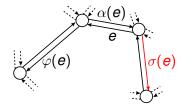
- ullet cycles de  $\alpha \longleftrightarrow$  arêtes
- cycles de  $\varphi \longleftrightarrow$  faces
- cycles de  $\sigma \longleftrightarrow$  sommets



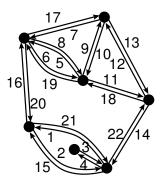
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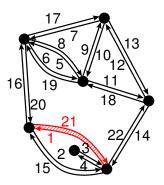


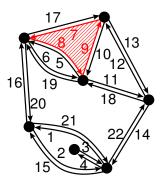
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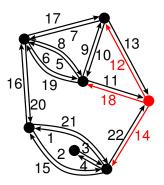


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# Graphes enrubannés

- ullet  $\alpha$ : involution sans point fixe
- $\varphi \alpha \sigma = 1$
- $\langle \varphi, \alpha, \sigma \rangle$  agit transitivement sur X

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# Exemple : comptage des arbres

- A<sub>n</sub>: arbres à n arêtes

$$A(x) := \sum_{t \in \mathcal{A}} x^{|t|}$$
$$= \sum_{n \in \mathbb{N}} |\mathcal{A}_n| x^{n+1}$$



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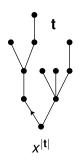
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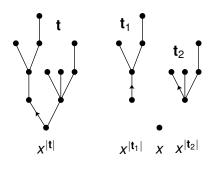
#### Exemple : comptage des arbres



- $\mathscr{A}_n$ : arbres à n arêtes
- \[ \alpha \]: arbres

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#### Comptage des cartes

#### Théorème

Le nombre de cartes planaires enracinées à n arêtes est

$$\frac{2}{n+2}$$
 3<sup>n</sup>Cat<sub>n</sub>.

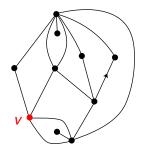


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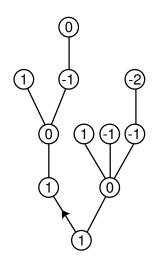
## Quadrangulations pointées





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## Arbres bien étiquetés



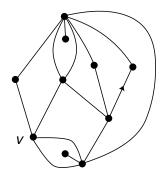
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## Comptage des cartes

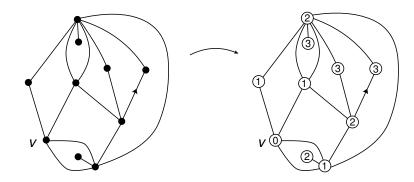
#### Théorème

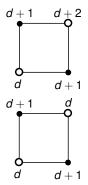
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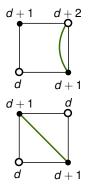
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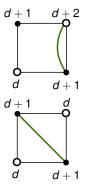


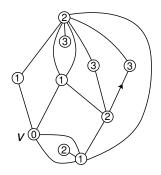


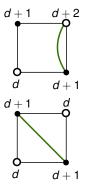


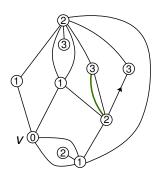


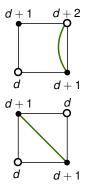
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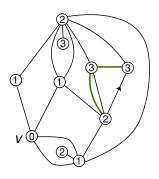


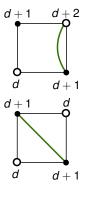


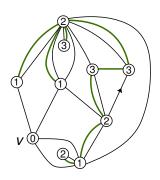


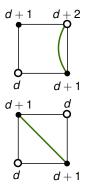


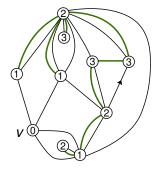


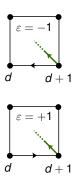


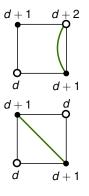


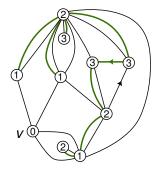


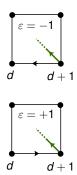


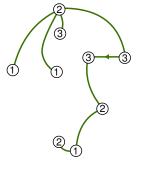




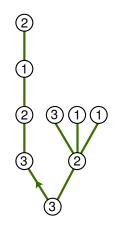


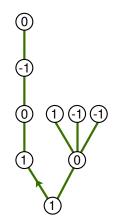


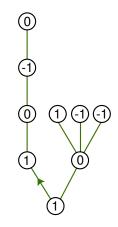


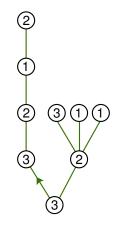


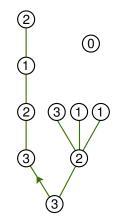
$$\varepsilon = +\mathbf{1}$$

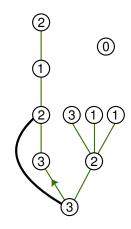


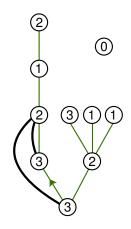




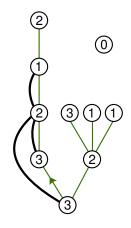


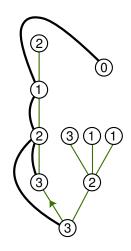


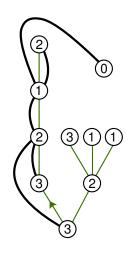


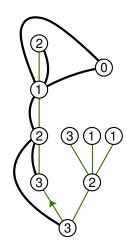


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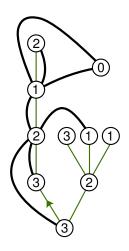


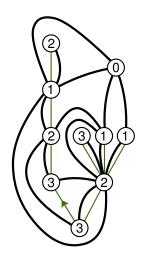


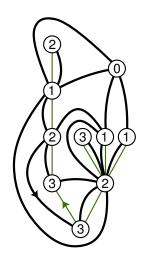




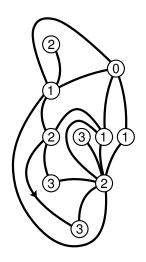
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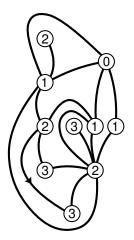


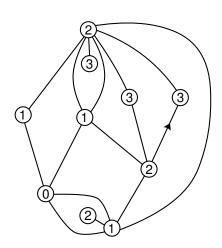




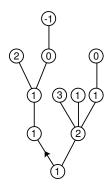
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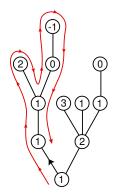




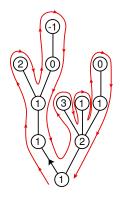
#### Processus de contour et d'étiquettes



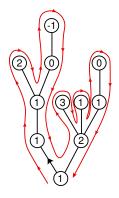
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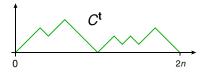


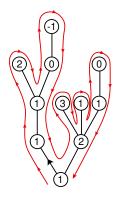


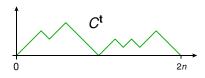














- $(\mathbf{t}_n, I_n)$  uniforme
- $C^n := C^{\mathbf{t}_n}$  et  $L^n := L^{\mathbf{t}_n}$

#### Théorème (Chassaing, Marckert, Schaeffer)

$$\left(\left(\frac{C_{2ns}^n}{(2n)^{\frac{1}{2}}}\right)_{0\leq s\leq 1}, \left(\frac{L_{2ns}^n}{\left(\frac{8n}{9}\right)^{\frac{1}{4}}}\right)_{0\leq s\leq 1}\right) \xrightarrow[n\to\infty]{(loi)} (\mathbf{e},Z),$$

pour la topologie uniforme sur  $\mathscr{C}([0,1],\mathbb{R})^2$ .

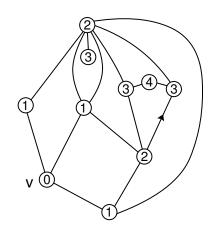
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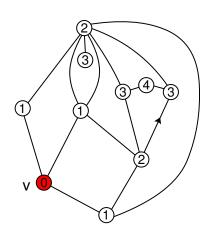
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• 
$$\mathcal{I}_{\mathbf{q},v}(0) = 1$$

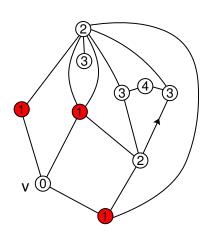
• 
$$I_{q,v}(1) = 3$$

• 
$$I_{q,v}(2) = 2$$

• 
$$\mathcal{I}_{q,v}(3) = 3$$

• 
$$I_{q,v}(4) = 1$$

• 
$$\mathcal{R}_{q,v} = 4$$



• 
$$\mathcal{I}_{\mathbf{q},\nu}(0) = 1$$

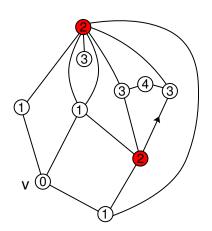
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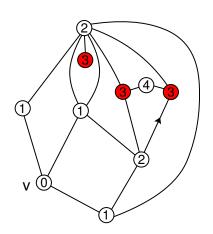
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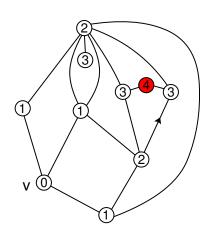
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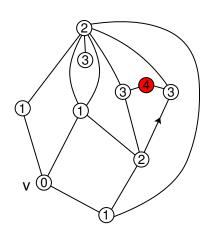
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• 
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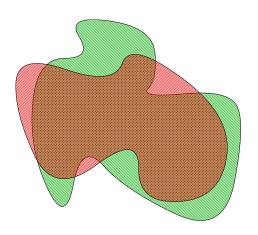
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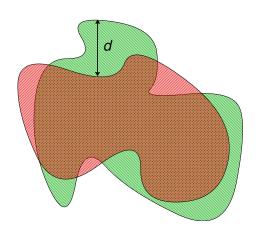
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### Distance de Hausdorff



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# Topologie de Gromov-Hausdorff

- [X, d]: classe d'isomorphismes de (X, d)
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L'espace métrique  $(V(\mathbf{q}_n), n^{-1/4}d_{gr})$  tend en loi pour la topologie de Gromov-Hausdorff, le long d'une sous-suite, vers un espace métrique aléatoire limite, noté (S, D).

#### Théorème (LE GALL)

La dimension de Hausdorff de (S, D) est p.s. 4

#### Théorème (LE GALL, PAULIN)

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